"El saber de mis hijos hará mi grandeza"

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Itô vs. Stratonovich: which calculus is better? The case of a Malthusian population model.

T E S I S

Que para obtener el grado académico de:

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"Jóvenes del mundo, piensen... en lo que sea... pero piensen."

Papa Juan XXIII

Con esta frase es que, en un encuentro con jóvenes en el vaticano, comienza el Papa Juan XXIII su emotivo sermón. Ésta frase ha sido una de mis más grandes motivaciones, a la hora de decidir estudiar un posgrado en matemáticas. Por tal motivo, quiero dedicar unas palabras a todas aquellas personas que, de una u otra forma, a su manera, me repitío esta frase, quizás sin saber ellos, pero logró que en momentos de debilidad no me rindiera sino, al contrario, me brindó esperanzas para continuar.

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Cesar Alberto Rosales Alcantar

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Con cariño para Fray Ivo Toneck.

Contents

Introduction

Differential equations are used to model the dynamics of many real systems. These equations are, in the majority, deterministic differential equations (ordinary or partial differential equations), where the main hypothesis on the model is that it only depends on relation between the state variables and the initial or boundary conditions. Even though, not all of the systems can be studied using such kind of models because they exhibit random fluctuations that have very important effects in their dynamics.

Differential equations that include random fluctuations are known as stochastic differential equations (SDE) and have been used to model systems in many fields as economics, finance and biology, among others. For instance, in biology, SDE are used to study epidemics [9], neuronal circuits [14], fishing models [2] and cell biology [5]. Likewise, SDE have been used to describe semiconductor manufacturing, for designing of chemical reactors and the sintering processes (production of ceramics) [10].

Most of these models are based in one of two stochastic calculus: Itô's calculus or Stratonovich's calculus. These are not the unique stochastic calculus [17], but they are the most popular. Some authors have made comparison between these stochastic calculus. For instance, Hodyss et al.[11] simulated atmospheric phenomenas using both calculus, Berkov and Gorn [1] showed that they lead to the same results in thermodynamics, Sancho [19] proved that Stratonovich calculus (slightly modified) is better to model colloidal particles, and Smythe et al.[20] showed the differences between those models with digital simulations.

Thus, one can wonder,

Which stochastic calculus should be used in mathematical models: Itô or Stratonovich?
Does it depend on the area of study?

In this thesis, we present the answer to this dilemma given by Braumann in [4] and [3] for the stochastic version of the general Malthusian population model given by

$$dN(t) = G(N(t))\,dt + \Sigma(N(t))\,dB(t)\,, \quad N(0) = N_0 > 0 \tag{1}$$

where $N(t)$ is the size of the population at time t, $G(x)$ is the average growth rate of the population, $\Sigma(x)$ the random fluctuations (sometimes due to the environment), and N_0 the initial population. It is important to note that the analysis of the model (1) will be realized in the scalar case.

Braumann [4] propose that the average growth rate is different when the system is studied by this stochastic calculus. With the Itô calculus, the so-called average growth rate is the arithmetic growth rate whereas with the Stratonovich calculus is the geometric growth rate.

Scheme of the thesis

This thesis is divided in 4 chapters. Chapter one presents some probability background, introduces the Brownian motion and discusses some of its sample-path properties. Next, Chapter two briefly introduces the two different integrals with respect to the Brownian motion in which we are interested, namely, the Itô integral and the Stratonovich integral.

Chapter three presents the theory of stochastic differential equation for population models in the scalar case. Some of the famous population growth models, like Verhulst's or Gompertz, do not satisfy the Itô conditions , so a existence and uniqueness theorem under different set of conditions is proved. The second part of this chapter consists to show the Feller criteria for explosions. The third part is dedicated to develop properties of the solution of a SDE, likewise the Markov property and diffusion characterization. Finally, this chapter ends with the boundaries classification of the states of the process.

In Chapter four, the Itô-Stratonovich dilemma is discussed for the Malthusian population model given in (1). The resolution of this dilemma is presented for the density-independent and the density-dependent cases. And finally, the dilemma is presented for the density-dependent harvesting case.

Chapter 1

Preliminaries

Introduction

This chapter is used to introduce the notation of the thesis. Also, the Brownian motion is presented and its relation between the Riemann-Stieltjes integral is exhibited.

1.1 Basic probability concepts and notation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The space $L^p(\Omega, \mathcal{F}, \mathbb{P})$ consists of all random variables X such that $\mathbb{E}[|X|^p] < \infty$. In particular, $L^1(\Omega, \mathcal{F}, \mathbb{P})$ is the set of integrable random variables.

A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ *converges almost surely* to the random variable (r.v.) X if there exists $N \in \mathcal{F}$ with $\mathbb{P}[N] = 0$ such that

$$X_n(\omega) \to X(\omega) \quad \forall \omega \in \Omega - N.$$

A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ *converges in* L^p to the r.v. X if, for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\mathbb{E}[|X_n - X|^p] < \epsilon \quad \text{for } n \geq N$$

If $p = 2$, this type of converge is known as mean-square convergence, and its denoted as ms-lim.

A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ *converges in probability* to the r.v. X if

$$\lim_{n \to \infty} \mathbb{P}[|X_n - X| > \epsilon] = 0 \ \forall \epsilon > 0.$$

Remark 1. The relation between these types of convergence is presented as follows:

(a) if $\{X_n\}_{n=1}^{\infty}$ converges to X almost surely, then $\{X_n\}_{n=1}^{\infty}$ converges to X in probability.

(b) if $\{X_n\}_{n=1}^{\infty}$ converges to X in L^p, then $\{X_n\}_{n=1}^{\infty}$ converges to X in probability.

(c) if $\{X_n\}_{n=1}^{\infty}$ converges to X in probability, there exists a subsequence $\{X_{n_k}\}_{k=1}^{\infty}$ that converges to X almost surely.

Let $T \subseteq \mathbb{R}$ be either $[a, b]$, with $0 \leq a < b$, or $[0, \infty)$. A *filtration* $\{\mathcal{F}_t\}_{t\in T}$ is a family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all indexes $s \leq t$ that belong to the set T. The collection $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \mathbb{P})$ is called *filtered space.*

A filtration $\{\mathcal{F}_t\}_{t\in T}$ is said to be *complete* if each sub-σ-algebra $\mathcal{F}_t$ contains all the P-null subsets. It is said to be *right-continuous* if

$$\mathcal{F}_t = \bigcap_{\substack{s>t \\ s\in T}} \mathcal{F}_s \ \ \forall t \in T.$$

If the filtration $\{\mathcal{F}_t\}_{t\in T}$ is complete and right-continuous, then it is said that satisfies the *usual conditions.*

A stochastic process $X(t), t \in T$ is a collection of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $\omega \in \Omega$, the mapping

$$t \to X(t, \omega), \ \ t \in T,$$

is called *trajectory* or *sample-path* of the stochastic process. It is said that the stochastic process $X(t), t \in T$ is *continuous* if almost all trajectories are continuous. Moreover, the process is said to be *adapted* to a filtration $\{\mathcal{F}_t\}_{t\in T}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each $t \in T$. Clearly, $X(t), t \in T$ is adapted to the so-called natural filtration given by $\mathcal{F}_t = \sigma(X_s | s \leq t)$, $t \in T$.

A stochastic process $M(t), t \in T$ is a *martingale* with respect to a filtration $\{\mathcal{F}_t\}_{t\in T}$ if

(i) $M(t), t \in T$, is adapted to $\{\mathcal{F}_t\}_{t\in T}$,

(ii) $M(t)$ is integrable for all $t \in T$, and

(iii) $\mathbb{E}\left[M(t) \middle| \mathcal{F}_s\right] = M(s) \ \forall s \leq t$ a.s.

In the case we hold $\leq$ $(\geq)$ indeed the equality in (iii), we called *sub-martingale* (*super-martingale*).

Recall that a partition π of an interval $T = [a, b]$ is a finite collection of points $\{t_0, t_1, \ldots, t_n\}$ such that

$$t_0 = a < t_1 < \cdots < t_{n-1} < t_n = b.$$

Let $\mathcal{P}(T)$ be the set of all finite partitions of T. The norm (or mesh) of a partition π is defined as

$$\| \pi \| := \max_{1\leq j\leq n} |t_j - t_{j-1}|.$$

The *total variation* of a stochastic process $X(t), t \in T$ is defined as

$$V_T(X) := \sup_{\pi\in\mathcal{P}(T)} \sum_{k=1}^{n} |X(t_k) - X(t_{k-1})|,$$

and the *quadratic variation* as

$$[X]_T := \operatorname*{P\text{-}lim}_{\|\pi\|\to 0} \sum_{k=1}^{n} |X(t_k) - X(t_{k-1})|^2$$

where P-lim means limit in probability.

1.2 Brownian motion

Definition 1.1. A stochastic process $B(t), t \in [0, \infty)$ with states in $\mathbb{R}$ is called a *standard Brownian motion* if it satisfies the next assumptions:

(a) $P[\omega \in \Omega | B(0, \omega) = 0] = 1$;

(b) for any $0 \leq s < t$, the random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$. In other words,

$$P[a \leq B(t) - B(s) \leq b] = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{x^2}{2(t-s)}} dx; \tag{1.1}$$

(c) the process has independent increments: for any $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables $B(t_1)$, $B(t_2) - B(t_1)$,$\cdots$, $B(t_n) - B(t_{n-1})$ are independent;

(d) Almost all sample-paths of $B(\cdot, \omega)$ are continuous functions, i.e.,

$$P[\omega \in \Omega | B(\cdot, \omega) \text{ is continuous }] = 1. \tag{1.2}$$

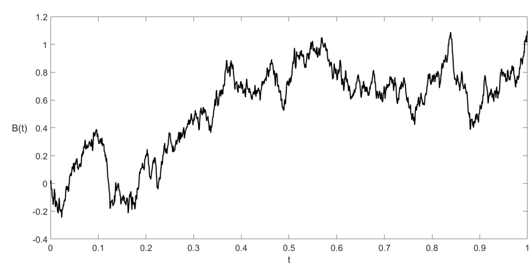

Figure 1.1: A Brownian motion sample-path.

The next remark collects some basic properties of brownian motion. The proofs can be found in [15, 21].

Remark 2. 1. $B(t) \sim N(0, t)$ $\forall t \geq 0$

2. $\mathbb{E}\left[|B(t) - B(s)|^2\right] = t - s$,

3. $\mathbb{E}\left[|B(t) - B(s)|^4\right] = 3|t - s|^2$,

4. for any $s, t \geq 0$, $\mathbb{E}[B(t) B(s)] = \min\{s, t\}$.

5. almost surely, $B(t, \omega)$ is an uniformly continuous function on finite intervals.

Let $\{\mathcal{B}_t\}_{t \in [a,b]}$ be the natural filtration of the Brownian motion $B(t), t \in [a, b]$.

Proposition 1.2. *The stochastic processes $B(t)$ and $B(t)^2 - t, t \in [a, b]$, are martingales with respect to the filtration $\{\mathcal{B}_t\}_{t \in [a,b]}$.*

Proof. By the definition of $\{\mathcal{B}_t\}_{t\in[a,b]}$, it follows that $B(t)$ and $B(t)^2 - t$ are $\mathcal{B}_t$-measurables for each $t \in [a,b]$. Obviously, $B(t)$ and $B(t)^2 - t$ are integrable for each t.

Let consider $a \leq s \leq t \leq b$. Because of the independent increments property of the Brownian motion $B(t)$ $a \leq t \leq b$, $B(t) - B(s)$ is independent of $\mathcal{B}_s$, implying that

$$\begin{aligned}\mathbb{E}\left[B(t)\middle|\mathcal{B}_s\right] &= \mathbb{E}\left[B(t) - B(s)\middle|\mathcal{B}_s\right] + \mathbb{E}\left[B(s)\middle|\mathcal{B}_s\right]\\ &= \mathbb{E}\left[B(t) - B(s)\right] + B(s)\\ &= B(s).\end{aligned}$$

With this, is evidently that $B(t), t \in [a,b]$ is a martingale with respect to $\{\mathcal{B}_t\}_{t\in[a,b]}$. On the other hand, $[B(t) - B(s)]^2$ is independent to $\mathcal{B}_s$ and,

$$\begin{aligned}\mathbb{E}\left[B(t)^2\middle|\mathcal{B}_s\right] &= \mathbb{E}\left[B(s)^2 + 2B(s)\left[B(t) - B(s)\right] + \left[B(t) - B(s)\right]^2\middle|\mathcal{B}_s\right]\\ &= \mathbb{E}\left[B(s)^2\middle|\mathcal{B}_s\right] + 2\mathbb{E}\left[B(s)\left[B(t) - B(s)\right]\middle|\mathcal{B}_s\right]\\ &\quad+\mathbb{E}\left[\left[B(t) - B(s)\right]^2\middle|\mathcal{B}_s\right]\\ &= B(s)^2 + 2B(s)\,\mathbb{E}\left[B(t) - B(s)\middle|\mathcal{B}_s\right] + \mathbb{E}\left[\left[B(t) - B(s)\right]^2\right]\\ &= B(s)^2 + t - s.\end{aligned}$$

Therefore, $B(t)^2 - t, t \in [a,b]$, is also a martingale with respect to $\{\mathcal{B}_t\}_{t\in[a,b]}$. □

Lemma 1.3. *Let $B(t), t \in [a,b]$ be a Brownian motion. Then,*

$$\underset{\|\pi\|\to 0}{\text{ms-lim}} \sum_{i=1}^{n} |B(t_i) - B(t_{i-1})|^2 = b - a.$$

Proof. Let $\pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$, let consider the random variable

$$X_n = \sum_{i=1}^{n} |B(t_i) - B(t_{i-1})|^2 - (b-a),$$

Note that

$$\begin{aligned}X_n^2 &= \left[\sum_{i=1}^{n} |B(t_i) - B(t_{i-1})|^2 - (b-a)\right]^2\\ &= \left\{\sum_{i=1}^{n} \left[|B(t_i) - B(t_{i-1})|^2 - (t_i - t_{i-1})\right]\right\}^2\\ &= \sum_{i=1}^{n} V_i^2 + 2\sum_{i<j} V_i V_j,\end{aligned}$$

where $V_i = |B(t_i) - B(t_{i-1})|^2 - (t_i - t_{i-1})$. Then,

$$\begin{aligned}\mathbb{E}[V_i] &= \mathbb{E}\left[|B(t_i) - B(t_{i-1})|^2\right] - (t_i - t_{i-1}) \\ &= (t_i - t_{i-1}) - (t_i - t_{i-1}) = 0.\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[V_i^2\right] &= \mathbb{E}\left[|B(t_i) - B(t_{i-1})|^4\right] - 2(t_i - t_{i-1})\mathbb{E}\left[|B(t_i) - B(t_{i-1})|^2\right] \\ &\quad + (t_i - t_{i-1})^2 \\ &= 3(t_i - t_{i-1})^2 - 2(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2 \\ &= 2(t_i - t_{i-1})^2.\end{aligned}$$

Now, let $\mathcal{G}_i := \sigma(V_j \mid j \leq i)$ be the σ-algebra generated by the random variables V_j for $j \leq i$. Then,

$$\begin{aligned}\mathbb{E}[V_i V_j] &= \mathbb{E}\left[\mathbb{E}\left[V_i V_j \middle| \mathcal{G}_i\right]\right], \ i < j \\ &= \mathbb{E}\left[V_i \mathbb{E}\left[V_j \middle| \mathcal{G}_i\right]\right], \ i < j \\ &= \mathbb{E}[V_i \mathbb{E}[V_j]] = 0.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\left[X_n^2\right] &= \mathbb{E}\left[\sum_{i=1}^{n} |B(t_i) - B(t_{i-1})|^2 - (b-a)\right]^2 \\ &= \sum_{i=1}^{n} \mathbb{E}\left[V_i^2\right] + 2\sum_{i<j} \mathbb{E}[V_i V_j] \\ &= 2\sum_{i=1}^{n} (t_i - t_{i-1})^2 \\ &\leq 2 \parallel \pi \parallel \sum_{i=1}^{n} (t_i - t_{i-1}) \\ &= 2 \parallel \pi \parallel (b-a),\end{aligned}$$

which implies that

$$\lim_{\|\pi\| \to 0} \mathbb{E}\left[\sum_{\pi} |\Delta B(t)|^2 - (b-a)\right]^2 = 0.$$

□

Corollary 1.4. *The quadratic variation of Brownian motion $B(t), t \in [a,b]$ is $b-a$, that is,*

$$[B]_{[a,b]} = \underset{\|\pi\| \to 0}{\text{P-lim}} \sum_{i=1}^{n} |B(t_i) - B(t_{i-1})|^2 = b - a.$$

Proof. By Remark 1(b), convergence in L^p implies convergence in probability. Then, Lemma 1.3 implies that

$$[B]_{[a,b]} = b - a.$$

□

Theorem 1.5. *Almost surely, the sample-paths of Brownian motion have unbounded total variation on any finite interval $[a, b]$.*

Proof. Suppose that the statement is not true, that is, there exists $\tilde{\Omega} \subseteq \Omega$ such that $P\left(\tilde{\Omega}\right) > 0$ and

$$V_{[a,b]}(B)(\omega) < \infty \; \forall \omega \in \tilde{\Omega}. \tag{1.3}$$

Consider $\Omega^\star = \{\omega \in \Omega | B(\cdot, \omega) \text{ is continuous } \}$ and $\omega \in \tilde{\Omega} \cap \Omega^\star$. As $B(t, \omega)$ is uniformly continuous P-a.s. in $[a, b]$, for each $\epsilon > 0$ there exists $\delta > 0$ such that if $s, t \in [a, b]$ then $|t - s| < \delta$ implies

$$|B(t, \omega) - B(s, \omega)| < \frac{\epsilon}{1 + V_{[a,b]}(B)(\omega)}.$$

Let π be a partition of $[a, b]$ such that $\| \pi \| < \delta$. Then,

$$\max_{\pi} |\Delta B|(\omega) := \max_{1 \le i \le n} |B(t_i, \omega) - B(t_{i-1}, \omega)| < \frac{\epsilon}{1 + V_{[a,b]}(B)(\omega)}. \tag{1.4}$$

On the other hand,

$$\begin{aligned}
\sum_{\pi} |\Delta B|^2(\omega) &:= \sum_{i=1}^{n} |B(t_i, \omega) - B(t_{i-1}, \omega)|^2 \\
&\le \left[\max_{1 \le i \le n} |B(t_i, \omega) - B(t_{i-1}, \omega)|\right] \sum_{i=1}^{n} |B(t_i, \omega) - B(t_{i-1}, \omega)| \\
&\le \max_{\pi} |\Delta B|(\omega) V_{[a,b]}(B)(\omega) \\
&\le \frac{V_{[a,b]}(B)(\omega)}{1 + V_{[a,b]}(B)(\omega)} \epsilon \\
&\le \epsilon,
\end{aligned}$$

which implies that

$$\lim_{\|\pi\| \to 0} \sum_{\pi} |\Delta B|^2(\omega) \mathbb{I}_{\tilde{\Omega}} = 0, \tag{1.5}$$

The latter equality leaves that

$$\text{P-}\lim_{\|\pi\| \to 0} \sum_{\pi} |\Delta B|^2(\omega) \mathbb{I}_{\tilde{\Omega}} = 0, \tag{1.6}$$

contradicting Corollary 1.4 which states that

$$\text{P-}\lim_{\|\pi\| \to 0} \sum_{\pi} |\Delta B|^2(\omega) \mathbb{I}_{\tilde{\Omega}} = b - a. \tag{1.7}$$

Therefore, the initial assumption is not true. Hence,

$$V_{[a,b]}(B) = \infty \text{ a.s.} \tag{1.8}$$

□

Relation between the Riemann-Stieltjes integral

Let $C([a,b])$ be the class of continuous functions $f(t)$ defined on $[a,b]$ with values on $\mathbb{R}$. For each $f,g \in C([a,b])$ and $\pi \in \mathcal{P}$, define $U(f,g,\pi)$ and $L(f,g,\pi)$ as

$$\begin{aligned} U(f,g,\pi) &= \sum_{i=1}^{n} f(t_i)\left(g(t_i) - g(t_{i-1})\right), \\ L(f,g,\pi) &= \sum_{i=1}^{n} f(t_{i-1})\left(g(t_i) - g(t_{i-1})\right). \end{aligned}$$

A function $g \in C([a,b])$ is called a *Riemann-Stieltjes integrator* if, for each $f \in C([a,b])$,

$$\lim_{\|\pi\|\to 0} \left|U(f,g,\pi) - L(f,g,\pi)\right| = 0, \tag{1.9}$$

and denote this class of functions by $\mathcal{I}$. Next, we present a characterization of the elements of $\mathcal{I}$ in terms of the total variation. The proof can be found in [6].

Proposition 1.6. *If $g \in \mathcal{I}$, then its total variation is finite, that is,*

$$V_{[a,b]}(g) := \sup_{\pi\in\mathcal{P}([a,b])} \sum_{k=1}^{n} \left|g(t_k) - g(t_{k-1})\right| < \infty.$$

Note that this result can be stated equivalently as follows: if $V_{[a,b]}(g) = \infty$ for any function, then $g \notin \mathcal{I}$.

Since the sample-paths of the Brownian motion $B(t), t \in [a,b]$ are continuous a.s., Theorem 1.5 and Proposition 1.6 show that the sample-paths $B(\cdot,\omega) \notin \mathcal{I}$ almost surely and can not be used as Riemann-Stieltjes integrators.

Therefore, there exists (at least) a stochastic process $Y(t), t \in [a,b]$ such that contradicts the definition of Riemann-Stieltjes integrator for $g(t) = B(t)$. One of these stochastic processes is the Brownian motion itself.

Proposition 1.7. *Let $B(t), t \in [a,b]$ be a Brownian motion. Then, there exists a sequence of partitions $\{\pi_n\}_{n=1}^{\infty}$ of $[a,b]$ such that $\| \pi_n \| \leq \frac{1}{n}$ and*

$$\lim_{n\to\infty} \left|U(B,B,\pi_n) - L(B,B,\pi_n)\right| = b - a, \text{ almost surely.}$$

Proof. First observe that

$$\begin{aligned} L(B,B,\pi) &= \sum_{i=1}^{n} B(t_{i-1})\left[B(t_i) - B(t_{i-1})\right], \\ U(B,B,\pi) &= \sum_{i=1}^{n} B(t_i)\left[B(t_i) - B(t_{i-1})\right]. \end{aligned}$$

Then,

$$\begin{aligned} U(B,B,\pi) - L(B,B,\pi) &= \sum_{i=1}^{n} \left[B(t_i) - B(t_{i-1})\right]^2 \xrightarrow{L^2} b-a, \\ U(B,B,\pi) + L(B,B,\pi) &= \sum_{i=1}^{n} \left[B(t_i)^2 - B(t_{i-1})^2\right] = b^2 - a^2, \end{aligned}$$

and Corollary 1.4 implies that,

$$\underset{\|\pi\|\to 0}{\text{P-lim}} \left| U(B,B,\pi) - L(B,B,\pi) \right| = b-a.$$

Let $\{\pi_n\}_{n=1}^{\infty} \subset \mathcal{P}$ be a sequence of partitions such that $\| \pi_n \| \leq \frac{1}{n}$. Using the Chebyshev inequality and straightforward calculus, we have that

$$\begin{aligned} P\left[\left|U(B,B,\pi_n) - L(B,B,\pi_n) - (b-a)\right| > \epsilon\right] &\leq \frac{\mathbb{E}\left[\left|U(B,B,\pi_n) - L(B,B,\pi_n) - (b-a)\right|^2\right]}{\epsilon^2} \\ &\leq \frac{2(b-a)\| \pi_n \|}{\epsilon^2} \\ &\leq \frac{2(b-a)}{\epsilon^2 n} \end{aligned}$$

Thus,

$$\underset{n\to\infty}{\text{P-lim}} \left| U(B,B,\pi_n) - L(B,B,\pi_n) \right| = b-a, .$$

Finally, Remark 1(c) implies the existence of a subsequence $\{\pi_{n_k}\}_{k=1}^{\infty}$ of $\{\pi_n\}_{n=1}^{\infty}$ such that

$$U(B,B,\pi_{n_k}) - L(B,B,\pi_{n_k}) \to b-a \text{ almost surely.}$$

□

Chapter 2

Stochastic Integrals

Introduction

It was shown in Chapter 1 that the sample-path of Brownian motion can not be used as Riemann-Stieltjes integrators. Thus, in order to define the integral of a stochastic process $f(t), t \in [a,b]$, with respect to the Brownian motion, say,

$$\int_a^b f(t,\omega)\, dB(t,\omega) \tag{2.1}$$

it is necessary to specify the time points in the partition at which the process is evaluated and the type of convergence used for computing the limit of the Riemann-Stieltjes sums.

In this chapter, we discuss first how such an integral can be defined and take as starting point the computation of the quantity

$$I(B(t), t \in [a,b], \alpha) := \underset{\|\pi\|\to 0}{\text{ms-lim}} \sum_{j=1}^{n} B((1-\alpha)t_{j-1} + \alpha t_j)\left[B(t_j) - B(t_{j-1})\right],$$

where $\alpha \in [0,1]$. The result of this computation will show light on how to choose α depending on the properties we want the stochastic integral satisfies.

Later, section 2.2 are dedicated to construct the Itô integral and the Itô calculus. Moreover, it is also shown, under certain conditions, that the Itô integral can be obtained as the limit of Riemann-Stieltjes sums in the mean-square or in probability convergence.

Finally, section 2.3 shows the construction of the Stratonovich integral using the Itô integral and some of its properties.

2.1 The importance of α.

In this section, the importance of the choose of α is explained by the computation of $I(B(t), t \in [a,b], \alpha)$.

Consider a partition π_n of $[a,b]$ with $t_i - t_{i-1} = \frac{b-a}{n}$, $\tau_j = (1-\alpha)t_{j-1} + \alpha t_j$ and define the auxiliary random variables:

- $\Delta B_{1-\alpha,j} = B(\tau_j) - B(t_{j-1}) \sim \mathcal{N}(0, \tau_j - t_{j-1})$,

- $\Delta B_{\alpha,j} = B(t_j) - B(\tau_j) \sim \mathcal{N}(0, t_j - \tau_j)$,
- $\Delta\Delta B_j^2 = \Delta B_{1-\alpha,j}^2 - \Delta B_{\alpha,j}^2$.

The next two lemmas show some properties of these random variables, which are used to show the mean-square convergence. Lemma 2.1 gives the first and second moment of the random variable $\Delta\Delta B_j^2$.

Lemma 2.1. *The Brownian motion $B(t), t \in [a,b]$ satisfies the equalities*

$$\begin{aligned}
\mathbb{E}\left[\Delta\Delta B_j^2\right] &= (2\alpha - 1)(t_j - t_{j-1}) \\
\mathbb{E}\left[\left(\Delta\Delta B_j^2\right)^2\right] &= \left\{2\left[\alpha^2 + (1-\alpha)^2\right] + (2\alpha-1)^2\right\}(t_j - t_{j-1})^2 .
\end{aligned}$$

Proof. The first equality follows from direct computations:

$$\begin{aligned}
\mathbb{E}\left[\Delta\Delta B_j^2\right] &= \mathbb{E}\left[\Delta B_{1-\alpha,j}^2 - \Delta B_{\alpha,j}^2\right] \\
&= \mathbb{E}\left[\Delta B_{1-\alpha,j}^2\right] - \mathbb{E}\left[\Delta B_{\alpha,j}^2\right] \\
&= (\tau_j - t_{j-1}) - (t_j - \tau_j) \\
&= (2\alpha - 1)(t_j - t_{j-1}) .
\end{aligned}$$

To prove the second equality, note that

$$\begin{aligned}
\mathbb{E}\left[\left(\Delta\Delta B_j^2\right)^2\right] &= \mathbb{E}\left[\left(\Delta B_{1-\alpha,j}^2 - \Delta B_{\alpha,j}^2\right)^2\right] \\
&= \mathbb{E}\left[\Delta B_{1-\alpha,j}^4 - 2\Delta B_{1-\alpha,j}^2 \Delta B_{\alpha,j}^2 + \Delta B_{\alpha,j}^4\right] \\
&= \mathbb{E}\left[\Delta B_{1-\alpha,j}^4\right] - 2\mathbb{E}\left[\Delta B_{1-\alpha,j}^2 \Delta B_{\alpha,j}^2\right] + \mathbb{E}\left[\Delta B_{\alpha,j}^4\right] \\
&= 3(\tau_j - t_{j-1})^2 - 2\mathbb{E}\left[\Delta B_{1-\alpha,j}^2\right]\mathbb{E}\left[\Delta B_{\alpha,j}^2\right] + 3(t_j - \tau_j)^2 \\
&= 3(\tau_j - t_{j-1})^2 - 2(\tau_j - t_{j-1})(t_j - \tau_j) + 3(t_j - \tau_j)^2 \\
&= 3\alpha^2(t_j - t_{j-1})^2 - 2\alpha(1-\alpha)(t_j - t_{j-1})^2 \\
&\quad +3(1-\alpha)^2(t_j - t_{j-1})^2 \\
&= \left\{2\left[\alpha^2 + (1-\alpha)^2\right] + (2\alpha-1)^2\right\}(t_j - t_{j-1})^2 .
\end{aligned}$$

□

Lemma 2.2. *The random variables $X_n := \sum_{j=1}^{n} \Delta\Delta B_j^2$, $n = 1, 2, \ldots$, satisfies the equalities*

$$\mathbb{E}[X_n] = (2\alpha - 1)(b - a) \tag{2.2a}$$

$$\mathbb{E}\left[X_n^2\right] = (2\alpha-1)^2(b-a)^2 + \frac{2\left[\alpha^2 + (1-\alpha)^2\right](b-a)^2}{n} \tag{2.2b}$$

Proof. First, the proof of (2.2a) is showed by noting that

$$\begin{aligned}
\mathbb{E}\left[X_n\right] &= \mathbb{E}\left[\sum_{j=1}^{n}\Delta\Delta B_j^2\right] = \sum_{j=1}^{n}\mathbb{E}\left[\Delta\Delta B_j^2\right] \\
&= \sum_{j=1}^{n}(2\alpha-1)\left(t_j-t_{j-1}\right) \\
&= (2\alpha-1)\sum_{j=1}^{n}\left(t_j-t_{j-1}\right) \\
&= (2\alpha-1)(b-a).
\end{aligned}$$

The proof of (2.2b) is as follows

$$\begin{aligned}
\mathbb{E}\left[X_n^2\right] &= \mathbb{E}\left[\left(\sum_{j=1}^{n}\left(\Delta\Delta B_j^2\right)\right)^2\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{n}\left(\Delta\Delta B_j^2\right)^2 + 2\sum_{1=i<j}^{n}\Delta\Delta B_j^2\Delta\Delta B_i^2\right] \\
&= \sum_{j=1}^{n}\mathbb{E}\left[\left(\Delta\Delta B_j^2\right)^2\right] + 2\sum_{1=i<j}^{n}\mathbb{E}\left[\Delta\Delta B_j^2\Delta\Delta B_i^2\right] \\
&= \sum_{j=1}^{n}\left\{2\left[\alpha^2+(1-\alpha)^2\right]+(2\alpha-1)^2\right\}\left(t_j-t_{j-1}\right)^2 \\
&\quad +2\sum_{1=i<j}^{n}\mathbb{E}\left[\Delta\Delta B_j^2\right]\mathbb{E}\left[\Delta\Delta B_i^2\right] \\
&= \sum_{j=1}^{n}\left\{2\left[\alpha^2+(1-\alpha)^2\right]+(2\alpha-1)^2\right\}\left(t_j-t_{j-1}\right)^2 \\
&\quad +2(2\alpha-1)^2\sum_{1=i<j}^{n}\left(t_j-t_{j-1}\right)\left(t_i-t_{i-1}\right) \\
&= 2\left[\alpha^2+(1-\alpha)^2\right]\sum_{j=1}^{n}\left(t_j-t_{j-1}\right)^2+(2\alpha-1)^2\sum_{1=i,j}^{n}\left(t_j-t_{j-1}\right)\left(t_i-t_{i-1}\right) \\
&= 2\left[\alpha^2+(1-\alpha)^2\right]\sum_{j=1}^{n}\left(\frac{b-a}{n}\right)^2+(2\alpha-1)^2\sum_{1=i,j}^{n}\left(\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \\
&= 2\left[\alpha^2+(1-\alpha)^2\right]\frac{(b-a)^2}{n}+(2\alpha-1)^2(b-a)^2.
\end{aligned}$$

□

The next lemma shows the mean-square convergence of the random variables X_n.

Lemma 2.3. *The sequence* $\{X_n\}_{n=1}^{\infty}$ *converges to* $(2\alpha-1)(b-a)$ *in* $L^2(\Omega)$*, that is,*

$$\underset{n\to\infty}{\text{ms-lim}}\, X_n = (2\alpha-1)(b-a).$$

Proof. Put $X=(2\alpha-1)(b-a)$ and observe that

$$\begin{aligned}
\mathbb{E}\left[|X_n-X|^2\right] &= \mathbb{E}\left[X_n^2-2XX_n+X^2\right]\\
&= \mathbb{E}\left[X_n^2\right]-2X\mathbb{E}\left[X_n\right]+X^2\\
&= 2\left[\alpha^2+(1-\alpha)^2\right]\frac{(b-a)^2}{n}+(2\alpha-1)^2(b-a)^2\\
&\quad -2(2\alpha-1)^2(b-a)^2+\left[(2\alpha-1)(b-a)\right]^2\\
&= 2\left[\alpha^2+(1-\alpha)^2\right]\frac{(b-a)^2}{n}.
\end{aligned}$$

Then, taking limit in the last equation, it follows that

$$\lim_{n\to\infty}\mathbb{E}\left[|X_n-X|^2\right]=\lim_{n\to\infty}2\left[\alpha^2+(1-\alpha)^2\right]\frac{(b-a)^2}{n}=0,$$

which proves our claim. □

Now, we are ready to give an explicit expression of $I(B(t),t\in[a,b],\alpha)$ in terms of the Brownian motion and α.

Theorem 2.4. *For each* $\alpha\in[0,1]$ *and interval* $[a,b]$*, it holds that*

$$I\Big(B(t),t\in[a,b],\alpha\Big)=\frac{1}{2}\left[B(b)^2-B(a)^2+(2\alpha-1)(b-a)\right]. \tag{2.3}$$

Proof. Recall that $\tau_j=(1-\alpha)t_{j-1}+\alpha t_j$ for $j=1,\dots,n$. Then

$$\begin{aligned}
I\Big(B(t),t\in[a,b],\alpha\Big) &= \underset{n\to\infty}{\text{ms-lim}}\sum_{j=1}^{n}B(\tau_j)\Big[B(t_j)-B(t_{j-1})\Big] \text{ with } \tau_j=(1-\alpha)t_{j-1}+\alpha t_j\\
&= \underset{n\to\infty}{\text{ms-lim}}\sum_{j=1}^{n}\Big\{B(\tau_j)\left[B(\tau_j)-B(t_{j-1})\right]+B(\tau_j)\left[B(t_j)-B(\tau_j)\right]\Big\}\\
&= \underset{n\to\infty}{\text{ms-lim}}\sum_{j=1}^{n}\frac{1}{2}\Big\{-\Big[B(\tau_j)-\big(B(\tau_j)-B(t_{j-1})\big)\Big]^2+\Big[B(\tau_j)-B(t_{j-1})\Big]^2\\
&\quad+\Big[B(\tau_j)+\big(B(t_j)-B(\tau_j)\big)\Big]^2-\Big[B(t_j)-B(\tau_j)\Big]^2\Big\}\\
&= \underset{n\to\infty}{\text{ms-lim}}\sum_{j=1}^{n}\frac{1}{2}\{B(t_j)^2-B(t_{j-1})^2\\
&\quad+\Big[B(\tau_j)-B(t_{j-1})\Big]^2-\Big[B(t_j)-B(\tau_j)\Big]^2\Big\}\\
&= \frac{1}{2}\underset{n\to\infty}{\text{ms-lim}}\sum_{j=1}^{n}\left\{\left[B(\tau_j)-B(t_{j-1})\right]^2-\left[B(t_j)-B(\tau_j)\right]^2\right\}\\
&\quad+\frac{1}{2}\left[B(b)^2-B(a)^2\right].
\end{aligned}$$

By Lemma 2.3, we have that

$$\underset{n\to\infty}{\text{ms-lim}} \sum_{j=1}^{n} \left\{ \left[B(\tau_j) - B(t_{j-1}) \right]^2 - \left[B(t_j) - B(\tau_j) \right]^2 \right\} = (2\alpha - 1)(b-a).$$

Thus,

$$I\Big(B(t), t \in [a,b], \alpha\Big) = \frac{1}{2}\left[B(b)^2 - B(a)^2\right] + \frac{1}{2}(2\alpha - 1)(b-a) \tag{2.4}$$

□

Since $I\Big(B(t), t \in [a,b], \alpha\Big)$ depends on α, two natural questions arise:

1. For which α the fundamental theorem of calculus is true?
2. For which α, the collection of variables

$$M_s := I\Big(B(t), t \in [a,s], \alpha\Big), s \in [a,b]$$

   is a martingale?

Concerning to the first question, notice that $\alpha = \frac{1}{2}$ gives the answer because, Theorem 2.4 leaves to

$$I\left(B(t), t \in [a,b], \frac{1}{2}\right) = \frac{1}{2}\left[B(b)^2 - B(a)^2\right].$$

Concerning to the second question, consider the natural filtration of the Brownian motion $\mathcal{B}_t, t \in [a,b]$. Using Proposition 1.2, we see that

$$\begin{aligned}
\mathbb{E}\left[M_t | \mathcal{B}_s\right] &= \mathbb{E}\left[\frac{1}{2}\left[B(t)^2 - B(a)^2 + (2\alpha-1)(t-a)\right]\Big|\mathcal{B}_s\right] \\
&= \frac{1}{2}\mathbb{E}\left[B(t)^2 + (2\alpha-1)t\Big|\mathcal{B}_s\right] - \frac{1}{2}\mathbb{E}\left[B(a)^2 + (2\alpha-1)a\Big|\mathcal{B}_s\right] \\
&= \frac{1}{2}\left[B(s)^2 + (t-s) + (2\alpha-1)t - B(a)^2 - (2\alpha-1)a\right] \\
&= \frac{1}{2}\left[B(s)^2 - B(a)^2 + (2\alpha-1)(s-a) + 2\alpha(t-s)\right] \\
&= M_s + 2\alpha(t-s).
\end{aligned}$$

Then, the value $\alpha = 0$ answers the question.

Remark 3. Let $Y(s) = B(b), s \in [a,b]$ and

$$\begin{aligned}
M(t) &= I\Big(Y(s), s \in [a,t], \alpha\Big), t \in [a,b] \\
&= B(b)\left[B(t) - B(a)\right] \text{ for } t \in [a,b]
\end{aligned}$$

Notice that this process is not adapted to the filtration $\{\mathcal{B}_t\}_{t\in[a,b]}$ and hence, $M(t), t \in [a,b]$ can not be a martingale. Thus, to ensure the martingale property is required the "integrand" processes have to be an adapted process to $\{\mathcal{B}_t\}_{t\in[a,b]}$.

Summarizing, we have the following:

- the value $\alpha = 0$ yields the process (2.4) is a martingale;
- remark 3 shows that the martingale property can fail if the integrand is not adapted to the filtration $\{\mathcal{B}_\mathrm{t}\}_{\mathrm{t}\in[\mathrm{a},\mathrm{b}]}$;
- the value $\alpha = \frac{1}{2}$ shows the fundamental theorem of calculus holds for the integrand $B(t), t \in [a, b]$.

The integrals resulting for $\alpha = 0$ is called Itô integral, whereas $\alpha = \frac{1}{2}$ is known as Stratonovich integral. The next sections give the rigorous definition of these integrals.

2.2 Itô Integral

This section discusses the construction of the Itô integral, given by Kuo ([15], Chapters 4 and 5). For that purpose, consider a Brownian motion $B(t), t \in [a, b]$ and a filtration $\{\mathcal{F}_t\}_{t\in[a,b]}$ satisfying the following conditions:

1. For each $t \in [a, b]$, $B(t)$ is $\mathcal{F}_t$-measurable.
2. For any $s \leq t$, the random variable $B(t) - B(s)$ is independent of the σ-algebra $\mathcal{F}_s$.

This construction is carried out in three steps. The first one define the integral for simple process, which are introduced below.

Definition 2.5. A stochastic process $f(t, \omega), t \in [a, b]$, adapted to $\{\mathcal{F}_t\}_{t\in[a,b]}$ is called *simple process* (or step process) if

$$f(t, \omega) = \sum_{j=1}^{n} \psi_{j-1}(\omega) \mathbb{I}_{[t_{j-1}, t_j)}(t), \tag{2.5}$$

where $\{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$, ψ_{k-1} are $\mathcal{F}_{t_{k-1}}$-measurable and ψ_k is a square integrable random variable for $k = 0, 1, \ldots, n-1$.

The second step extends the integral to square-integrable stochastic process, which are defined as follows.

Definition 2.6. A stochastic process $f(t, \omega)$ adapted to $\{\mathcal{F}_t\}_{t\in[a,b]}$ is called *square-integrable* if

$$\int_a^b \mathbb{E}\left[|f(t)|^2\right] dt < \infty. \tag{2.6}$$

This step is realized by a mean-square limit of simple process. The third step extends the integral to stochastic process that satisfies the condition

$$\int_a^b |f(t)|^2 dt < \infty, \text{ a.s.} \tag{2.7}$$

In other words, the third step consists in the construction of an stochastic integral for process $f(t,\omega)$ whose sample paths are functions in $L^2([a,b])$. This construction is necessary because not all continuous process are square integrable; for example, $X(t) = \exp\left[B(t)^2\right]$ does not satisfy (2.6) but it does condition (2.7).

Step 1: simple process

Denote by $S_{ad}([a,b])$ the set of simple processes adapted to the filtration $\{\mathcal{F}_t\}_{t\in[a,b]}$ on $[a,b]$. For $f(t)\in S_{ad}$, define the *Itô integral* of f as the Riemann-Stieltjes sum

$$\int_a^b f(s)dB(s) := \sum_{k=1}^{n} \psi_{k-1}\left[B(t_k) - B(t_{k-1})\right]. \tag{2.8}$$

The Itô integral $\int_a^b \cdot dB(s)$ satisfies the below properties,

1. $\mathbb{E}\left[\int_a^b f(s)dB(s)\right] = 0.$
2. $\mathbb{E}\left[\left|\int_a^b f(s)dB(s)\right|^2\right] = \int_a^b \mathbb{E}\left[f(t)^2\right]ds.$
3. $\int_a^t f(s)dB(s)$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in[a,b]}$.

Property 2 implies that the mapping

$$f(s) \to \int_a^b f(s)dB(s)$$

is an isometry from $S_{ad}[a,b]$ into $L^2(\Omega)$.

The next lemma is important to extend the stochastic integral to square-integrable process.

Lemma 2.7. *Let $f(t,\omega)$ a simple process in $S_{ad}([a,b])$. Then, the inequality*

$$P\left\{\left|\int_a^b f(s)dB(s)\right| > \epsilon\right\} \leq \frac{C}{\epsilon^2} + P\left\{\int_a^b \left|f(t)\right|^2 dt > C\right\}$$

holds for any positive constants ϵ and C.

Proof. For each $C>0$, define the stochastic process

$$f_C(t,\omega) = \begin{cases} f(t,\omega) & \text{if } \int_a^t |f(s,\omega)|^2 ds \leq C \\ 0 & \text{otherwise} \end{cases}$$

Next, observe that

$$\left\{\left|\int_a^b f(s)dB(s)\right| > \epsilon\right\} \subset \left\{\left|\int_a^b f_C(s)dB(s)\right| > \epsilon\right\} \cup \left\{\int_a^b f(s)dB(s) \neq \int_a^b f_C(s)dB(s)\right\}.$$

which implies that

$$P\left\{\left|\int_a^b f(s)dB(s)\right|>\epsilon\right\}\leq P\left\{\left|\int_a^b f_C(s)dB(s)\right|>\epsilon\right\}+P\left\{\int_a^b f(s)dB(s)\neq\int_a^b f_C(s)dB(s)\right\}.$$

On the other hand, note that

$$\left\{\int_a^b f(s)dB(s)\neq\int_a^b f_C(s)dB(s)\right\}\subset\left\{\int_a^b |f(t)|^2\,dt>C\right\}$$

and also that $\int_a^b |f_C(t)|^2 dt \leq C$ a.s., so $\mathbb{E}\left[\int_a^b |f_C(t)|^2 dt\right]\leq C$. Then,

$$\begin{aligned}
P\left\{\left|\int_a^b f(s)dB(s)\right|>\epsilon\right\} &\leq P\left\{\left|\int_a^b f_C(s)dB(s)\right|>\epsilon\right\}+P\left\{\int_a^b f(s)dB(s)\neq\int_a^b f_C dB(s)\right\}\\
&\leq P\left\{\left|\int_a^b f_C(s)dB(s)\right|>\epsilon\right\}+P\left\{\int_a^b |f(t)|^2\,dt>C\right\}\\
&\leq \frac{1}{\epsilon^2}\mathbb{E}\left[\left|\int_a^b f_C(s)dB(s)\right|^2\right]+P\left\{\int_a^b |f(t)|^2\,dt>C\right\}\\
&= \frac{1}{\epsilon^2}\int_a^b \mathbb{E}\left[|f_C(t)|^2\right]dt+P\left\{\int_a^b |f(t)|^2\,dt>C\right\}\\
&\leq \frac{C}{\epsilon^2}+P\left\{\int_a^b |f(t)|^2\,dt>C\right\},
\end{aligned}$$

which is the desired result. □

Step 2: square integrable process

Let $L^2_{ad}([a,b]\times\Omega)$ be the set of square-integrable processes, that is, the set of stochastic processes that satisfy (2.6). It is clear that $S_{ad}\subset L^2_{ad}([a,b]\times\Omega)$. The next lemma shows that $S_{ad}[a,b]$ is dense in $L^2_{ad}([a,b]\times\Omega)$.

Lemma 2.8. *Let $f\in L^2_{ad}([a,b]\times\Omega)$. There exists a sequence $\{f_n(t)\}_{n=1}^{\infty}\subset S_{ad}([a,b])$ such that*

$$\lim_{n\to\infty}\int_a^b \mathbb{E}\left[f(t)-f_n(t)\right]^2 dt=0. \tag{2.9}$$

Proof. The proof of this lemma is given in three cases.

- *First case.* In this case we prove the lemma assuming that the mapping

  $$(s,t)\to\mathbb{E}\left[f(s)\,f(t)\right] \tag{2.10}$$

  is continuous on $[a,b]\times[a,b]$.

  Let $\pi_n=\{t_0,\ldots,t_n\}$ be a partition of $[a,b]$ and define the stochastic process $f_n(t,\omega)=f(t_{i-1},\omega)$, $t_{i-1}<t\leq t_i$. Note that $f_n(t,\omega)\in S_{ad}[a,b]$, for each n. The continuity of mapping in (2.10) implies that

  $$\lim_{s\to t}\mathbb{E}\left[|f(t)-f(s)|^2\right]=0;$$

  in particular,

  $$\lim_{n\to\infty}\mathbb{E}\left[|f(t)-f_n(t)|^2\right]=0. \tag{2.11}$$

On the other hand,

$$\begin{aligned}\mathbb{E}\left[|f(t)-f_n(t)|^2\right] &\leq 2\left(\mathbb{E}\left[f(t)^2\right]+\mathbb{E}\left[f_n(t)^2\right]\right)\\ &\leq 4\sup_{a\leq s\leq b}\mathbb{E}\left[f(s)^2\right] \qquad (2.12)\end{aligned}$$

Finally, properties (2.11) and (2.12) combined with the Lebesgue bounded convergence theorem imply that (2.9) is satisfied.

- *Second case.* Consider a bounded stochastic process $f \in L^2_{ad}([a,b]\times\Omega)$ and define

$$g_n(t,\omega) = \int_0^{n(t-a)} e^{-\tau} f\left(t-\frac{\tau}{n},\omega\right)d\tau$$

It is not difficult to prove that $g_n \in L^2_{ad}([a,b]\times\Omega)$. Moreover, for each n, g_n satisfies the following two properties:

(a) $\mathbb{E}[g_n(t)\,g_n(s)]$ is a continuous function in (s,t);

(b) $\int_a^b \mathbb{E}\left[|f(t)-g_n(t)|^2\right]dt \to 0$ as $n\to\infty$.

The fact (a) implies the existence of a simple process $f_n(t)$ such that

$$\int_a^b \mathbb{E}\left[|f_n(t)-g_n(t)|^2\right]dt \leq \frac{1}{n}. \qquad (2.13)$$

From (2.13) and the fact (b), it follows that

$$\begin{aligned}\int_a^b \mathbb{E}[f(t)-f_n(t)]^2\,dt &\leq 2\int_a^b \mathbb{E}[f(t)-g_n(t)]^2\,dt\\ &\quad +2\int_a^b \mathbb{E}[g_n(t)-f_n(t)]^2\,dt\\ &\leq 2\int_a^b \mathbb{E}[f(t)-g_n(t)]^2\,dt+\frac{2}{n},\end{aligned}$$

which in turn implies that

$$\lim_{n\to\infty}\int_a^b \mathbb{E}[f(t)-f_n(t)]^2\,dt = 0.$$

- *Third case.* Consider a process $f \in L^2_{ad}([a,b]\times\Omega)$ and define

$$g_n(t,\omega) = \begin{cases} f(t,\omega) & \text{if} \quad |f(t,\omega)| \leq n,\\ 0 & \text{if} \quad |f(t,\omega)| > n,\end{cases}$$

for each n. Now, as g_n is bounded, the second case of this proof implies the existence of a simple process $f_n(t)$ such that

$$\int_a^b \mathbb{E}\left[|f_n(t)-g_n(t)|^2\right]dt \leq \frac{1}{n};$$

thus, the Lebesgue dominated convergence theorem implies that

$$\int_a^b \mathbb{E}\left[|f(t)-g_n(t)|^2\right]dt \to 0,$$

which complete the proof.

□

Proposition 2.9. *Let $\{f_n(t)\}_{n=1}^{\infty}$ be a sequence of step process on $S_{ad}([a,b])$ such that satisfies Lemma 2.8 for $f(t)\in L^2_{ad}([a,b]\times\Omega)$. Then, the sequence*

$$\left\{\int_a^b f_n(s)dB(s)\right\}_{n=1}^{\infty} \tag{2.14}$$

is a convergent sequence in $L^2(\Omega)$.

Proof. Since the sequence $\{f_n(t)\}_{n=1}^{\infty}\subset S_{ad}([a,b])$ and $f_n-f_m\in S_{ad}([a,b])$ for each $n,m\in\mathbb{N}$, it follows that

$$\begin{aligned}
\mathbb{E}\left[\left|\int_a^b f_n(s)dB(s)-\int_a^b f_m(s)dB(s)\right|^2\right] &= \mathbb{E}\left[\left|\int_a^b \big(f_n(s)-f_m(s)\big)dB(s)\right|^2\right]\\
&= \int_a^b \mathbb{E}\left[|f_n(s)-f_m(s)|^2\right]ds
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}
\int_a^b \mathbb{E}\left[|f_n(s)-f_m(s)|^2\right]ds &= \int_a^b \mathbb{E}\left[|f_n(s)-f(s)+f(s)-f_m(s)|^2\right]ds\\
&\le 2\int_a^b \mathbb{E}\left[|f_n(s)-f(s)|^2\right]ds\\
&\quad +2\int_a^b \mathbb{E}\left[|f(s)-f_m(s)|^2\right]ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
\lim_{n,m\to\infty}\mathbb{E}\left[\left|\int_a^b f_n(s)dB(s)-\int_a^b f_m(s)dB(s)\right|^2\right] &\le \lim_{n,m\to\infty}\int_a^b \mathbb{E}\left[|f_n(t)-f_m(t)|^2\right]ds\\
&\le 2\lim_{n\to\infty}\int_a^b \mathbb{E}\left[|f_n(s)-f(s)|^2\right]ds\\
&\quad +2\lim_{m\to\infty}\int_a^b \mathbb{E}\left[|f(s)-f_m(s)|^2\right]ds = 0
\end{aligned}$$

Thus, the sequence (2.14) is a Cauchy sequence in $L^2(\Omega)$. Moreover, as $L^2(\Omega)$ is a complete metric space, every Cauchy sequence has a limit and thus,

$$L=\lim_{n\to\infty}\int_a^b f_n(s)dB(s)$$

exists. □

Let show that L does not depend of the choice of the sequence. For instance, let $\{f_n(t)\}_{n=1}^{\infty}$ and $\{g_m(t)\}_{n=1}^{\infty}$ be two sequences on $S_{ad}([a,b])$ such that satisfies Lemma 2.8 for $f(t)$. Thus,

$$\begin{aligned}
\lim_{n,m\to\infty} \mathbb{E}\left[\left|\int_a^b f_n(s)dB(s) - \int_a^b g_m(s)dB(s)\right|^2\right] &= \lim_{n,m\to\infty} \mathbb{E}\left[\left|\int_a^b \big(f_n(s) - g_m(s)\big)dB(s)\right|^2\right] \\
&= \lim_{n,m\to\infty} \int_a^b \mathbb{E}\left[|f_n(s) - g_m(s)|^2\right]ds \\
&\leq 2\lim_{n,m\to\infty} \int_a^b \mathbb{E}\left[|f_n(s) - f(s)|^2\right]ds \\
&\quad +2\lim_{n,m\to\infty} \int_a^b \mathbb{E}\left[|g_m(s) - f(s)|^2\right]ds = 0.
\end{aligned}$$

Therefore,

$$L = \underset{n\to\infty}{\text{ms-lim}} \int_a^b f_n(s)dB(s) = \underset{m\to\infty}{\text{ms-lim}} \int_a^b g_m(s)dB(s).$$

By Proposition 2.9 and the last observation, the definition of the Itô integral for functions on $L^2_{ad}([a,b]\times\Omega)$, given below, is well-defined.

Definition 2.10. The Itô integral of a process $f(t,\omega) \in L^2_{ad}([a,b]\times\Omega)$ is defined as

$$\int_a^b f(s)dB(s) := \underset{n\to\infty}{\text{ms-lim}} \int_a^b f_n(s)dB(s) \tag{2.15}$$

where $\{f_n(s)\}_{n=1}^{\infty}$ is a sequence of simple processes in $S_{ad}([a,b])$ that satisfies Lemma 2.8 for $f(s)$.

Proposition 2.11. *Let be $f \in L^2_{ad}([a,b]\times\Omega)$. Then, $\int_a^t f(s)dB(s)$ is a martingale on t with respect to the filtration $\{\mathcal{F}_t\}_{t\in[a,b]}$.*

This proof can be found in [18].

Step 3: an extension by localization

Denote by $\mathcal{L}_{ad}\left(\Omega, L^2([a,b])\right)$ the class of stochastic process $f(t,\omega), t \in [a,b]$ adapted to $\{\mathcal{F}_t\}$ with almost all sample paths in $L^2([a,b])$. More briefly,

$$\mathcal{L}_{ad}\left(\Omega, L^2([a,b])\right) = \{f(t,\omega) \mid f(t,\omega) \text{ satisfy (2.7) a.s.}\} \tag{2.16}$$

It is not difficult to show that

$$L^2_{ad}([a,b]\times\Omega) \subset \mathcal{L}_{ad}\left(\Omega, L^2([a,b])\right).$$

Moreover,

$$\mathcal{L}_{ad}\left(\Omega, L^2[a,b]\right) - L^2_{ad}([a,b]\times\Omega) \neq \emptyset.$$

The next example shows the last assertion.

Example 1. Consider $[a,b] = [0,1]$ and let be $f(t) = \exp\{B(t)^2\}$. After some calculations, one can obtain that

$$\mathbb{E}\left[|f(t)|^2\right] = \mathbb{E}\left[\exp\{2B(t)^2\}\right] = \begin{cases} (1-4t)^{-\frac{1}{2}} & \text{if} \quad 0 \leq t < \frac{1}{4}, \\ \infty & o.c. \quad . \end{cases}$$

Hence,

$$\int_0^1 \mathbb{E}\left[|f(t)|^2\right] dt = \infty,$$

which means that

$$f(t) \notin L^2_{ad}([0,1] \times \Omega).$$

On the other hand, $f(t)$ is uniformly continuous a.s. on $[0,1]$. This implies that exists a constant $M > 0$ such that $|f(t)| \leq M$ a.s.. Then, $\int_0^1 |f(t)|^2 dt < M^2 < \infty$ a.s., that is,

$$f(t) \in \mathcal{L}_{ad}\left(\Omega, L^2[0,1]\right).$$

In order to define the Itô integral for a stochastic process $f \in \mathcal{L}_{ad}\left(\Omega, L^2[a,b]\right)$, we need two approximations lemmas. The first one implies that $L^2_{ad}([a,b] \times \Omega)$ is dense on $\mathcal{L}_{ad}\left(\Omega, L^2[a,b]\right)$ with respect to the convergence in probability.

Lemma 2.12. *Let $f \in \mathcal{L}_{ad}\left(\Omega, L^2([a,b])\right)$. Then there exists a sequence $\{f_n(t,\omega)\}$ in $L^2_{ad}([a,b] \times \Omega)$ such that*

$$\lim_{n\to\infty} \int_a^b |f_n(t) - f(t)|^2 dt = 0$$

almost surely and in probability.

Proof. For each n, let define

$$f_n(t,\omega) = \begin{cases} f(t,\omega) & \text{if} \int_a^t |f(s,\omega)|^2 ds \leq n, \\ 0 & \text{o.c.} \end{cases}$$

The stochastic process $f_n(t,\omega)$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\in[a,b]}$. Also,

$$\int_a^b |f_n(t,\omega)|^2 dt = \int_a^{\tau_n(\omega)} |f(t,\omega)|^2 dt \quad \text{a.s.},$$

where $\tau_n(\omega) := \sup\{\, t \ : \ \int_a^t |f(s,\omega)|^2 ds \leq n \,\}$. Thus,

$$\int_a^b |f_n(t,\omega)|^2 dt \leq n \quad \text{a.s.}.$$

Then, $\int_a^b \mathbb{E}\left[|f_n(t,\omega)|^2\right] dt \leq n$ and $f_n \in L^2_{ad}([a,b] \times \Omega)$.

By other hand, for each $\omega \in \Omega$ there exists $N = N(\omega) \in \mathbb{N}$ such that

$$\int_a^b |f(t,\omega)|^2 dt < N;$$

thus, $f_N(t,\omega) = f(t,\omega)$ for all $t \in [a,b]$. Therefore,

$$\lim_{n\to\infty} \int_a^b |f_n(t,\omega) - f(t,\omega)|^2 dt = 0 \text{ a.s.}$$

Finally, the convergence in probability is obtained by the Remark 1(a). □

The next lemma shows that $S_{ad}[a,b]$ is dense on $\mathcal{L}_{ad}$ with respect to the convergence in probability.

Lemma 2.13. *Let $f(t) \in \mathcal{L}_{ad}\left(\Omega, L^2[a,b]\right)$. Then there exists a sequence $\{f_n(t)\}_{n=1}^{\infty}$ of simple process of $S_{ad}([a,b])$ such that*

$$\lim_{n\to\infty} \int_a^b |f_n(t) - f(t)|^2 \, dt = 0 \text{ in probability.}$$

Proof. As $f(t) \in \mathcal{L}_{ad}\left(\Omega, L^2[a,b]\right)$, Lemma 2.12 implies the existence of a sequence $\{g_n(t)\}_{n=1}^{\infty} \subset L^2_{ad}([a,b]\times\Omega)$ such that

$$\lim_{n\to\infty} \int_a^b |g_n(t) - f(t)|^2 \, dt = 0 \text{ in probability.}$$

By Lemma 2.8, for each $g_n(t)$ there exists a step process $f_n(t) \subset S_{ad}[a,b]$ such that

$$\mathbb{E}\left[\int_a^b \left|f_n(t) - g_n(t)\right|^2 dt\right] < \frac{1}{n}.$$

Next, if $\epsilon > 0$, then

$$\left\{\int_a^b \left|f_n(t) - f(t)\right|^2 dt > \epsilon\right\} \subset \left\{\int_a^b \left|f_n(t) - g_n(t)\right|^2 dt > \frac{\epsilon}{4}\right\} \cup \left\{\int_a^b \left|g_n(t) - f(t)\right|^2 dt > \frac{\epsilon}{4}\right\}.$$

Thus,

$$\begin{aligned} P\left\{\int_a^b \left|f_n(t) - f(t)\right|^2 dt > \epsilon\right\} &\le P\left\{\int_a^b \left|f_n(t) - g_n(t)\right|^2 dt > \frac{\epsilon}{4}\right\} \\ &\quad + P\left\{\int_a^b \left|g_n(t) - f(t)\right|^2 dt > \frac{\epsilon}{4}\right\} \\ &\le \frac{4}{\epsilon n} + P\left\{\int_a^b \left|g_n(t) - f(t)\right|^2 dt > \frac{\epsilon}{4}\right\} \end{aligned}$$

and the proof is completed. □

Proposition 2.14. *Let $f(t) \in \mathcal{L}_{ad}\left(\Omega, L^2[a,b]\right)$ and $\{f_n(t)\}_{n=1}^{\infty} \subset S_{ad}[a,b]$ the sequence of step processes that satisfies Lemma 2.13. Then, the sequence*

$$\left\{\int_a^b f_n(s) dB(s)\right\}_{n=1}^{\infty} \tag{2.17}$$

is convergent in probability.

Proof. By Lemma 2.7, if $h = f_n - f_m$, $\epsilon > 0$ and $C = \frac{\epsilon^3}{2}$, then

$$\begin{aligned} P\left\{\left|\int_a^b h(s)dB(s)\right| > \epsilon\right\} &= P\left\{\left|\int_a^b \Big(f_n(s) - f_m(s)\Big)dB(s)\right| > \epsilon\right\} \\ &\leq \frac{\epsilon}{2} + P\left\{\int_a^b \Big|f_n(t) - f_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\}. \end{aligned}$$

Also,

$$\left\{\int_a^b \Big|f_n(t) - f_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} \subset \left\{\int_a^b \Big|f_n(t) - f(t)\Big|^2 dt > \frac{\epsilon^3}{8}\right\} \cup \left\{\int_a^b \Big|f(t) - f_m(t)\Big|^2 dt > \frac{\epsilon^3}{8}\right\}$$

and

$$\begin{aligned} P\left\{\int_a^b \Big|f_n(t) - f_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} &\leq P\left\{\int_a^b \Big|f_n(t) - f(t)\Big|^2 dt > \frac{\epsilon^3}{8}\right\} \\ &\quad + P\left\{\int_a^b \Big|f(t) - f_m(t)\Big|^2 dt > \frac{\epsilon^3}{8}\right\}. \end{aligned}$$

This concludes that

$$\lim_{n,m\to\infty} P\left\{\int_a^b \Big|f_n(t) - f_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} = 0.$$

Thus, there exists $N \in \mathbb{N}$ such that

$$P\left\{\int_a^b \Big|f_n(t) - f_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} < \frac{\epsilon}{2} \ \forall\, n, m \geq N$$

which implies that

$$\begin{aligned} P\left\{\left|\int_a^b h(s)dB(s)\right| > \epsilon\right\} &= P\left\{\left|\int_a^b f_n(s)dB(s) - \int_a^b f_m(s)dB(s)\right| > \epsilon\right\} \\ &\leq \frac{\epsilon}{2} + P\left\{\int_a^b \Big|f_n(t) - f_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} \\ &\leq \epsilon \ \ \forall\, n, m \geq N. \end{aligned}$$

Finally, the last equation shows that the sequence (2.17) is convergent in probability and

$$L = \lim_{n\to\infty} \int_a^b f_n(s)dB(s) \ \text{ in probability}$$

exists. □

Let show that L does not depend of the choice of the sequence. For instance, let $\{f_n(s)\}_{n=1}^{\infty}$ and $\{g_m(s)\}_{n=1}^{\infty}$ be two sequences on $S_{ad}([a,b])$ that satisfies Lemma 2.13 for $f(s)$.

Applying Lemma 2.7 for $h = f_n - g_m$, $\epsilon > 0$ and $C = \frac{\epsilon^3}{2}$, we hold that

$$\begin{aligned} P\left\{\Big|\int_a^b h(s)dB(s)\Big| > \epsilon\right\} &= P\left\{\Big|\int_a^b \big(f_n(s) - g_m(s)\big)dB(s)\Big| > \epsilon\right\} \\ &\leq \frac{\epsilon}{2} + P\left\{\int_a^b \Big|f_n(t) - g_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\}. \end{aligned}$$

By other hand,

$$\left\{\int_a^b \Big|f_n(t) - g_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} \subset \left\{\int_a^b \Big|f_n(t) - f(t)\Big|^2 dt > \frac{\epsilon^3}{8}\right\} \cup \left\{\int_a^b \Big|f(t) - g_m(t)\Big|^2 dt > \frac{\epsilon^3}{8}\right\}$$

implies that

$$\begin{aligned} P\left\{\int_a^b \Big|f_n(t) - g_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} &\leq P\left\{\int_a^b \Big|f_n(t) - f(t)\Big|^2 dt > \frac{\epsilon^3}{8}\right\} \\ &\quad + P\left\{\int_a^b \Big|g_m(t) - f(t)\Big|^2 dt > \frac{\epsilon^3}{8}\right\}, \end{aligned}$$

which concludes, by Lemma 2.13, that

$$\lim_{n,m\to\infty} P\left\{\int_a^b \Big|f_n(t) - g_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} = 0$$

Thus, there exists $N \in \mathbb{N}$ such that

$$P\left\{\int_a^b \Big|f_n(t) - g_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} < \frac{\epsilon}{2} \ \ \forall\, n, m \geq N$$

which implies that

$$\begin{aligned} P\left\{\Big|\int_a^b h(s)dB(s)\Big| > \epsilon\right\} &= P\left\{\Big|\int_a^b f_n(s)dB(s) - \int_a^b g_m(s)dB(s)\Big| > \epsilon\right\} \\ &\leq \frac{\epsilon}{2} + P\left\{\int_a^b \Big|f_n(t) - g_m(t)\Big|^2 dt > \frac{\epsilon^3}{2}\right\} \\ &\leq \epsilon \ \ \forall\, n, m \geq N. \end{aligned}$$

Finally, the last equation shows that

$$L = \lim_{n\to\infty} \int_a^b f_n(s)dB(s) = \lim_{m\to\infty} \int_a^b g_m(s)dB(s) \ \text{ in probability.}$$

From the Proposition 2.14 and the last observation, the stochastic integral for processes in $\mathcal{L}_{ad}\left(\Omega, L^2([a,b])\right)$ is well-defined.

Definition 2.15. The Itô integral of a function $f(t) \in \mathcal{L}_{ad}\left(\Omega, L^2([a,b])\right)$ is defined as

$$\int_a^b f(s)dB(s) = \lim_{n\to\infty} \int_a^b f_n(s)dB(s) \ \text{ in probability,}$$

where $\{f_n(t)\}_{n=1}^{\infty}$ is a sequence of simple processes in $S_{ad}([a,b])$ such that

$$\lim_{n\to\infty} \int_a^b |f_n(t) - f(t)|^2 dt = 0 \ \text{ in probability.}$$

Riemann-Stieltjes sums

The Itô integral can be expressed by means of limit of Riemann-Stieltjes sums in the case of continuous stochastic process.

Theorem 2.16. *Suppose $f(t)$ is a continuous stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\in[a,b]}$. Then $f(t) \in \mathcal{L}_{ad}\left(\Omega, L^2([a,b])\right)$ and*

$$\int_a^b f(s)dB(s) = \lim_{\|\pi_n\|\to 0} \sum_{i=1}^n f(t_{i-1})\left[B(t_i) - B(t_{i-1})\right] \text{ in probability,}$$

where $\pi_n = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ is a partition of $[a,b]$ and $\| \pi_n \|$ is the partition norm. If in addition, $f(t) \in L^2_{ad}([a,b]\times\Omega)$ and $\mathbb{E}[f(t)f(s)]$ is a continuous function of t and s, then

$$\int_a^b f(s)dB(s) = \lim_{\|\pi_n\|\to 0} \sum_{i=1}^n f(t_{i-1})\left[B(t_i) - B(t_{i-1})\right] \text{ in } L^2(\Omega).$$

Proof. Let $\pi_n = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$ and

$$f_n(t,\omega) := \sum_{i=1}^n f(t_{i-1})\, 1_{[t_{i-1},t_i)}(t).$$

As f is a continuous stochastic process, then

$$\int_a^b |f_n(t) - f(t)|^2 dt \to 0$$

almost surely and in probability as $n\to\infty$. This implies that,

$$\int_a^b f(t)\,dB(t) = \lim_{||\pi_n||\to 0} \int_a^b f_n(t)\,dB(t) \quad \text{in probability.}$$

Moreover, as $f_n \in S_{ad}[a,b]$, then

$$\int_a^b f_n(t)\,dB(t) = \sum_{i=1}^n f(t_{i-1})\left[B(t_i) - B(t_{i-1})\right],$$

which implies that

$$\int_a^b f(t)\,dB(t) = \lim_{||\pi_n||\to 0} \sum_{i=1}^n f(t_{i-1})\left[B(t_i) - B(t_{i-1})\right] \quad \text{in probability,}$$

which proves the first assertion. In the case that $f \in L^2_{ad}([a,b]\times\Omega)$ and $\mathbb{E}[f(t)f(s)]$ is a continuous function of t and s, let $g_n(t,\omega) = f(t_i,\omega)$, $t_{i-1} < t \le t_i$. By the first case in the proof of Lemma 2.8, we have that equation (2.9) is satisfied and hence,

$$\int_a^b f(s)dB(s) = \lim_{n\to\infty} \int_a^b g_n(s)dB(s) \text{ in } L^2(\Omega).$$

Also,

$$\begin{aligned}\int_a^b g_n(s)dB(s) &= \sum_{i=1}^n f_n(t_{i-1})[B(t_i)-B(t_{i-1})] \\ &= \sum_{i=1}^n f(t_{i-1})[B(t_i)-B(t_{i-1})].\end{aligned}$$

Then, we conclude that

$$\int_a^b f(s)dB(s) = \lim_{n\to\infty}\sum_{i=1}^n f(t_{i-1})[B(t_i)-B(t_{i-1})] \text{ in } L^2(\Omega).$$

□

The Itô formula

This section begins with an example that illustrates one of the differences between the Riemann-Stieltjes integral and the Itô integral.

Since the identity function $f(t)=t, t\in\mathbb{R}$, satisfies the assumptions in Theorem 2.16, it follows from equation (2.4) with $\alpha=0$ that

$$\int_a^t B(s)dB(s) = \frac{1}{2}\left[B(t)^2 - B(a)^2 - (t-a)\right]$$

Thus, the Itô integral does not follows the usual rules of the Riemann-Stieltjes integral. Note that the above equality can be re-written as

$$B(t)^2 = B(a)^2 + 2\int_a^t B(s)dB(s) + (t-a),\ \forall\, t\in[a,b].$$

Now, putting $f(t)=t^2, t\in\mathbb{R}$, the above equality becomes in

$$f\big(B(t)\big) = f\big(B(a)\big) + \int_a^t f'\big(B(s)\big)dB(s) + \frac{1}{2}\int_a^t f''\big(B(s)\big)ds.$$

The so-called Itô formula shows that this formula holds for all functions $f\in C^2([a,b])$.

The Itô formula open the door to a stochastic calculus; thus, it can be considered as the "fundamental theorem of stochastic calculus".

Theorem 2.17. *If $f\in C^2([a,b])$, then*

$$f\big(B(t)\big) = f\big(B(a)\big) + \int_a^t f'\big(B(s)\big)dB(s) + \frac{1}{2}\int_a^t f''\big(B(s)\big)ds \tag{2.18}$$

where the first integral is an Itô integral and the second one is a Riemann integral for each sample path of $B(s)$.

The equality (2.18) is called Itô formula and its proof can be found in [15, p.95]. Here, we present an informal approach. Consider f as a C^2-function and observe that

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{1}{2} f''\big(x_0 + \lambda(x - x_0)\big)(x - x_0)^2$$

where $0 < \lambda < 1$. Therefore, if $\pi_n = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$, then

$$\begin{aligned} f\big(B(b)\big) - f\big(B(a)\big) &= \sum_{i=1}^{n} \Big[f\big(B(t_i)\big) - f\big(B(t_{i-1})\big) \Big] \\ &= \sum_{i=1}^{n} f'\big(B(t_{i-1})\big) \Big[B(t_i) - B(t_{i-1}) \Big] \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} f''\Big(B(t_{i-1}) + \lambda\big[B(t_i) - B(t_{i-1})\big] \Big) \Big[B(t_i) - B(t_{i-1}) \Big]^2. \end{aligned}$$

The first summation converges in probability to the Itô integral of $f'(B(t))$, that is,

$$\underset{\|\pi_n\| \to 0}{\text{P-lim}} \sum_{i=1}^{n} f'(B(t_{i-1}))[B(t_i) - B(t_{i-1})] = \int_a^b f' dB(s).$$

On the other hand, there exists a subsequence of partitions $\{\pi_{n_k}\}_{k=1}^{\infty} \subset \{\pi_n\}_{n=1}^{\infty}$ with $\| \pi_{n_k} \| \to 0$ as $k \to \infty$, such that the second summation converges to the Riemann integral of $f''(B(t))$, that is,

$$\sum_{i=1}^{n_k} f''\Big(B\big(t_{n_{i-1}}\big) + \lambda\big[B(t_{n_i}) - B\big(t_{n_{i-1}}\big)\big] \Big) \big[B(t_{n_i}) - B\big(t_{n_{i-1}}\big)\big]^2 \to \int_a^b f''(B(t))\, dt.$$

For each $f \in C^2([a, b])$, the Itô formula (2.18) represents the stochastic process $f(B(t))$ as a sum of a Riemann integral and an Itô integral. The stochastic processes that can be expressed in this form are called Itô processes.

Let $\mathcal{L}_{ad}\big(\Omega, L^1([a, b])\big)$ be the set of stochastic process $g(t, \omega)$ adapted to $\{\mathcal{F}_t\}_{t \in [a,b]}$, such that

$$P\left[\int_a^b \Big| g(t, \omega) \Big| dt < \infty \right] = 1.$$

Definition 2.18. Let $\{\mathcal{F}_t\}_{t \in [a,b]}$ a filtration that satisfies the two conditions expressed at the beginning of Section 2.2. A stochastic process $X(t, \omega)$ is called an *Itô process* if $X(a)$ is $\mathcal{F}_a$-measurable and there exists $f \in \mathcal{L}_{ad}\big(\Omega, L^2([a, b])\big)$ and $g \in \mathcal{L}_{ad}\big(\Omega, L^1([a, b])\big)$ such that

$$X(t) = X(a) + \int_a^t f(s)\, dB(s) + \int_a^t g(s)\, ds, \;\; a \le t \le b, \tag{2.19}$$

Equation (2.19) is called the integral form of an Itô process.

Remark 4. Equation (2.19) is also written as

$$dX(t) = f(t)\, dB(t) + g(t)\, dt, \tag{2.20}$$

and is called the "stochastic differential form" of an Itô process. This expression is quite useful to do straightforward calculus, but it does not has a formal meaning because the sample-paths of the Brownian motion are nowhere differentiable.

The next theorem presents a generalization of the Itô formula for Itô processes.

Theorem 2.19. *Let $X(t)$ be an Itô process given by*

$$X(t) = X(a) + \int_a^t f(s)\, dB(s) + \int_a^t g(s)\, ds, \;\; a \leq t \leq b.$$

Suppose $F(t,x)$ is a continuous function with continuous partial derivatives F_t, F_x and F_{xx}. Then, $F(t, X(t))$ is also an Itô process and

$$\begin{aligned} F(t, X(t)) &= F(a, X(a)) + \int_a^t F_x(s, X(s))\, f(s) dB(s) \\ &\quad + \int_a^t \Big[F_t(s, X(s)) + F_x(s, X(s))\, g(s) \\ &\quad + \frac{1}{2} F_{xx}(s, X(s))\, f(s)^2\Big]\, ds. \end{aligned} \tag{2.21}$$

Using the "stochastic differential form" of an Itô process, this generalization can be formally obtained using the Itô table:

$\times$	$dB(t)$	dt
$dB(t)$	dt	0
dt	0	0

If

$$dX(t) = f(t)\, dB(t) + g(t)\, dt,$$

then,

$$\begin{aligned} dF(t, X(t)) &= F_t(t, X(t))\, dt + F_x(t, X(t))\, dX(t) + \frac{1}{2} F_{xx}(t, X(t))\, (dX(t))^2 \\ &= F_t(t, X(t))\, dt + F_x(t, X(t)) \Big[f(t)\, dB(t) + g(t)\, dt\Big] \\ &\quad + \frac{1}{2} F_{xx}(t, X(t)) \Big[f(t)\, dB(t) + g(t)\, dt\Big]^2 \\ &= F_t(t, X(t))\, dt + F_x(t, X(t))\, f(t)\, dB(t) + F_x(t, X(t))\, g(t)\, dt \\ &\quad + \frac{1}{2} F_{xx}(t, X(t)) \Big[f(t)^2 (dB(t))^2 + f(t)\, g(t)\, dB(t)\, dt \\ &\quad + g(t)\, f(t)\, dt dB(t) + g(t)^2 (dt)^2\Big] \\ &= F_t(t, X(t))\, dt + F_x(t, X(t))\, f(t)\, dB(t) \\ &\quad + F_x(t, X(t))\, g(t)\, dt + \frac{1}{2} F_{xx}(t, X(t))\, f(t)^2\, dt \\ &= F_x(t, X(t))\, f(t)\, dB(t) + \Big[F_t(t, X(t)) \\ &\quad + F_x(t, X(t))\, g(t) + \frac{1}{2} F_{xx}(t, X(t))\, f(t)^2\Big]\, dt. \end{aligned}$$

2.3 Stratonovich integral

Recall that in Section 2.1 we saw that

$$I\left(B\left(t\right),t\in[a,b],\frac{1}{2}\right)=\frac{1}{2}\left[B\left(b\right)^{2}-B\left(a\right)^{2}\right].$$

This shows that, if one seek to define a stochastic integral that satisfies the usual rules of calculus, then $\alpha=\frac{1}{2}$ is necessary. This stochastic integral will be called Stratonovich integral, and it can be defined in terms of the Itô integral of an Itô process.

Definition 2.20. Let $X\left(t,\omega\right)$ be an Itô process given by

$$X\left(t\right)=X\left(a\right)+\int_{a}^{t}f\left(s\right)dB\left(s\right)+\int_{a}^{t}g\left(s\right)ds,\;\;a\leq t\leq b.$$

The Stratonovich integral of $X\left(t,\omega\right)$ with respect to the Brownian motion $B\left(t,\omega\right)$, denoted by $S\left(X,[a,b]\right)$, is defined by

$$\int_{a}^{b}X\left(s\right)\circ dB\left(s\right):=\int_{a}^{b}X\left(s\right)dB\left(s\right)+\frac{1}{2}\int_{a}^{b}f\left(s\right)ds. \tag{2.22}$$

In general, for an Itô process $X\left(t\right),t\in[a,b]$, there exists a modified version of the Itô formula for the Stratonovich integral. Let start with the next lemma.

Lemma 2.21. *Let $X\left(t\right)$ be an Itô process given by*

$$X\left(t\right)=X\left(a\right)+\int_{a}^{t}f\left(s\right)dB\left(s\right)+\int_{a}^{t}g\left(s\right)ds,\;\;a\leq t\leq b.$$

Suppose $F\left(t,x\right)$ is a continuous function with continuous partial derivative F_x and F_{xx}. Then,

$$\begin{aligned}\int_{a}^{t}F_{x}\left(s,X\left(s\right)\right)f\left(s\right)\circ dB\left(s\right) &= \int_{a}^{t}F_{x}\left(s,X\left(s\right)\right)f\left(s\right)dB\left(s\right)\\ &\quad+\frac{1}{2}\int_{a}^{t}F_{xx}\left(s,X\left(s\right)\right)f\left(s\right)^{2}ds.\end{aligned}$$

Then, the Itô formula for Stratonovich integral is presented here.

Theorem 2.22. *Let $X\left(t\right)$ be an Itô process given by*

$$X\left(t\right)=X\left(a\right)+\int_{a}^{t}f\left(s\right)dB\left(s\right)+\int_{a}^{t}g\left(s\right)ds,\;\;a\leq t\leq b.$$

Suppose $F\left(t,x\right)$ is a continuous function with continuous partial derivatives F_t, F_x and F_{xx}. Then, $F\left(t,X\left(t\right)\right)$ is also an Itô process and

$$\begin{aligned}F\left(t,X\left(t\right)\right) &= F\left(a,X\left(a\right)\right)+\int_{a}^{t}F_{x}\left(s,X\left(s\right)\right)f\left(s\right)\circ dB\left(s\right)\\ &\quad+\int_{a}^{t}\left[F_{t}\left(s,X\left(s\right)\right)+F_{x}\left(s,X\left(s\right)\right)g\left(s\right)\right]ds.\end{aligned}$$

The proof of this theorem is obtained applying the Itô formula (2.21) for the Itô process $X(t), t \in [a,b]$ and Lemma 2.21. In particular, for $X(t) = B(t)$ ($f(t) = 1$ and $g(t) = 0$), the Stratonovich integral satisfies the integration by parts formula.

Corollary 2.23. *Let $h(t,x)$ a continuous function, $H(t,x)$ an antiderivative in x of $h(t,x)$ and assume that H_t, h_t and h_x are continuous. Then,*

$$\int_a^b h(t, B(t)) \circ dB(s) = H(t, B(t))\Big|_a^b - \int_a^b H_t(s, B(s))\, ds. \tag{2.23}$$

In the case that $h(t,x)$ does not depend of t, then

$$\int_a^b h(t, B(t)) \circ dB(s) = H(t, B(t))\Big|_a^b. \tag{2.24}$$

Riemann-Stieltjes sums

The Stratonovich integral can be also obtained as a limit of Riemann-Stieltjes sums.

Theorem 2.24. *Suppose $f(t,x)$ is a continuous function with continuous partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$. Then,*

$$\begin{aligned}\int_a^b f(t, B(t)) \circ dB(s) &= \underset{\|\pi_n\|\to 0}{\text{P-lim}} \sum_{k=1}^{n} f\left(t_k^\star, \frac{1}{2}\left(B(t_{k-1}) + B(t_k)\right)\right)\left[B(t_k) - B(t_{k-1})\right] \\ &= \underset{\|\pi_n\|\to 0}{\text{P-lim}} \sum_{k=1}^{n} f\left(t_k^\star, B\left(\frac{t_{k-1}+t_k}{2}\right)\right)\left[B(t_k) - B(t_{k-1})\right]\end{aligned}$$

where $\pi_n = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ is a partition of $[a,b]$, $t_k^\star \in [t_{k-1}, t_k]$ is arbitrary and $\| \pi_n \|$ is the partition norm.

Chapter 3

Stochastic Differential Equations

Introduction

This chapter has the aim to introduce the concept of solution of a stochastic differential equation (SDE) (in the Itô sense) and to establish sufficient conditions about the existence and uniqueness of solutions for SDE that describe population models.

In section 3.1, the concept of solution of a SDE is presented, the theorem of existence and uniqueness of solutions with non-usual Itô conditions [8, p.48] is presented and an approximation theorem for their moments is exhibited. In section 3.2, the idea of a global solution for SDE is defined and we establish the feller criteria to avoid explosions for global solution ([12], [16]).

At section 3.3, the characterization as Markov and diffusion process of solution of SDE is showed and finally, in section 3.4 is dedicated to define the concept of attracting and attainable states [13].

3.1 Solution of stochastic differential equations

Let $B(t), t \in [a,b]$ be a Brownian motion on $[a,b]$ and $\{\mathcal{F}_t\}_{t\in[a,b]}$ a filtration such that $B(t)$ is $\mathcal{F}_t$-measurable and $B(t) - B(s)$ is independent of $\mathcal{F}_s$ for any $s \leq t$.

Definition 3.1. Let $F, G : [a,b] \times \mathbb{R} \to \mathbb{R}$ be $\mathcal{F}$-measurable functions. A stochastic process $X(t), t \in [a,b]$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a strong solution of the stochastic differential equation (SDE)

$$dX(t) = F(t, X(t))\, dB(t) + G(t, X(t))\, dt, \tag{3.1}$$

with initial condition $X(a) = Z$ if

(a) $X(t)$ is $\mathcal{F}_t$-measurable for each $t \in (a,b)$;

(b) $X(t), [a,b]$ is a continuous process;

(c) $P[X(a) = Z] = 1$;

(d) $F(t, X(t)) \in \mathcal{L}_{ad}\left(\Omega, L^2([a,b])\right)$, $G(t, X(t)) \in \mathcal{L}_{ad}\left(\Omega, L^1([a,b])\right)$; and,

(e) with probability 1,

$$X(t) = Z + \int_a^t F(s, X(s))\, dB(s) + \int_a^t G(s, X(s))\, ds, \quad a \leq t \leq b. \tag{3.2}$$

The functions F, G are called the coefficients of the SDE (3.1).

As in the theory of ordinary differential equations, there exists conditions for the coefficients F, G that guarantee the existence and uniqueness of a strong solution for the stochastic differential equation:

(a) **Lipschitz condition.** There exists a constant $K > 0$ such that

$$|F(t,x) - F(t,y)| + |G(t,x) - G(t,y)| \leq K|x-y|, \tag{3.3}$$

for all $t \in [a,b]$, $x, y \in \mathbb{R}$.

(b) **Growth condition.** There exists $K > 0$ such that

$$|F(t,x)|^2 + |G(t,x)|^2 \leq K\left(1 + x^2\right) \tag{3.4}$$

for all $t \in [a,b]$, $x \in \mathbb{R}$.

These conditions are called **Itô conditions**. The next result shows that the Itô conditions guarantee the existence and uniqueness of a strong solution for a stochastic differential equation. Its proof can be found in [8, p. 40].

Theorem 3.2. *Let $F, G : [a,b] \times \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying the Itô conditions and Z a r.v. independent of the Brownian motion such that $\mathbb{E}\left[|Z|^2\right] < \infty$. Then there exists a strong solution X of the SDE (3.1) such that*

$$\sup_{a \leq t \leq b} \mathbb{E}\left[X(t)^2\right] < \infty. \tag{3.5}$$

Moreover, if X_1 and X_2 are two strong solutions of (3.1), then

$$P\left[\sup_{a \leq t \leq b} |X_1(t) - X_2(t)| = 0\right] = 1. \tag{3.6}$$

Corollary 3.3. *Assume that the functions F and G satisfies the hypothesis of Theorem 3.2 and $X(t), t \in [a,b]$ is a strong solution of (3.1). Also, suppose that $\tilde{F}, \tilde{G} : [a,b] \times \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying the Itô conditions and that $\tilde{X}(t), t \in [a,b]$ is a strong solution of*

$$d\tilde{X}(t) = \tilde{F}\left(t, \tilde{X}(t)\right) dB(t) + \tilde{G}\left(t, \tilde{X}(t)\right) dt \tag{3.7}$$

If there exists some $N > 0$ such that

$$\tilde{F}(t,x) = F(t,x), \quad \tilde{G}(t,x) = G(t,x) \quad \text{for } |x| \leq N,\ t \in [a,b]$$

then

$$P\left[\sup_{a \leq t \leq \tau} |X(t) - \tilde{X}(t)| = 0\right] = 1,$$

where

$$\tau := \inf_{a \leq t \leq b} \{\max\left(|X(t)|, |\tilde{X}(t)|\right) > N\}.$$

Although the Itô conditions guarantee the existence and uniqueness of strong solutions for the SDE (3.1), there exists function that are used to describe population dynamics and not satisfies them. Remark 3 of [8, p.48] shows that the Lipschitz can be weakened to a local version and the growth condition can be changed as follows:

(a) **Local lipschitz condition.** For all $t \in [a,b]$ and each $N > 0$, there exists L_N such that

$$|F(t,x) - F(t,y)| + |G(t,x) - G(t,y)| \leq L_N |x-y| \tag{3.8}$$

for $(x,y) \in [-N,N] \times [-N,N]$.

(b) **Growth condition 2.** For all $t \in [a,b]$, $x \in \mathbb{R}$, there exists $K > 0$ such that

$$xG(t,x) + |F(t,x)|^2 \leq K\left(1 + x^2\right). \tag{3.9}$$

The next theorem shows that the above conditions guarantee the existence and uniqueness of strong solutions.

Theorem 3.4. *Let $F, G : [a,b] \times \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying the local Lipschitz condition (3.8), the growth condition (3.9) and Z a r.v. such that $\mathbb{E}\left[|Z|^2\right] < \infty$. Then (3.1) has a unique strong solution satisfying the initial condition Z and this solution is unique in the sense of Theorem 3.2.*

The proof of this result relies on the Gronwall-Bellman inequality given in the next lemma.

Lemma 3.5. *If $f(t)$ and $g(t)$ are functions in $L^1[a,b]$ and $\beta > 0$ such that*

$$g(t) \leq f(t) + \beta \int_a^t g(s)\, ds \ \ \forall\, t \in [a,b]. \tag{3.10}$$

Then,

$$g(t) \leq f(t) + \beta \int_a^t e^{\beta(t-s)} f(s)\, ds \ \ \forall\, t \in [a,b] \tag{3.11}$$

In particular, if $f \equiv \alpha \in \mathbb{R}$, then

$$g(t) \leq \alpha e^{\beta(t-a)} \ \ \forall\, t \in [a,b]$$

Proof. In order to prove this lemma, we first show that

$$\int_a^t g(s)\, ds \leq \int_a^t e^{\beta(t-s)} f(s)\, ds \ \text{ almost everywhere.}$$ [1]

To do this, define

$$h(t) = \int_a^t g(s)\, ds \ \ \forall\, t \in [a,b].$$

[1] *almost everywhere* convergece is used with respect to the Lebesgue measure.

By (3.10),

$$h'(t) - \beta h(t) \leq f(t) \text{ almost surely.}$$

Next, multiplying both sides of the last equation by $e^{\beta t}$, we obtain

$$e^{\beta t}\left[h'(t) - \beta h(t)\right] \leq e^{\beta t} f(t) \text{ almost everywhere.}$$

It easy to note that the left side is $\frac{d}{dt}\left[e^{\beta t} h(t)\right]$. Then,

$$e^{\beta t} h(t) \leq \int_a^t e^{\beta s} f(s)\, ds \text{ almost everywhere.}$$

Thus,

$$h(t) \leq \int_a^t e^{\beta(t-s)} f(s)\, ds \text{ almost everywhere.}$$

Now, by (3.10), it follows that

$$\begin{aligned} g(t) &\leq f(t) + \beta \int_a^t g(s)\, ds, \;\; \forall\, t \in [a,b] \\ &\leq f(t) + \beta \int_a^t e^{\beta t - s} f(s)\, ds, \;\; \forall\, t \in [a,b] \end{aligned}$$

as we want to prove. □

Let proceed to prove Theorem 3.4.

Proof. We first show the existence of such a solution. Let consider

$$Z_N = \begin{cases} Z & \text{if } |Z| \leq N, \\ \frac{Z}{|Z|} N & \text{if } |Z| > N; \end{cases}$$

$$G_N(t,x) = \begin{cases} G(t,x) & \text{if } |x| \leq N, \\ G\left(t, \frac{x}{|x|} N\right) & \text{if } |x| > N; \end{cases}$$

$$F_N(t,x) = \begin{cases} F(t,x) & \text{if } |x| \leq N, \\ F\left(t, \frac{x}{|x|} N\right) & \text{if } |x| > N; \end{cases}$$

The functions $F_N(t,x), G_N(t,x)$ and the random variable Z_N satisfy the conditions in Theorem 3.2. Thus, there is a unique strong solution $X_N(t), t \in [a,b]$ to the stochastic differential equation

$$dX_N(t) = F_N(t, X_N(t))\, dB(t) + G_N(t, X_N(t))\, dt \tag{3.12}$$

with initial condition $X_N(a) = Z_N$. Next, let

$$\tau_N := \sup\left\{ t \in [a,b] \mid \sup_{a \leq s \leq t} |X_N(s)| \leq N \right\} \tag{3.13}$$

and consider $M > N$. As

$$F_M(t,x) = F_N(t,x), \quad G_M(t,x) = G_N(t,x), \quad \text{for } x \in [-N,N], \tag{3.14}$$

by the Corollary 3.3, $X_N(t) = X_M(t)$ in probability on $[a, \tau_N]$. Then,

$$P\left[\sup_{a\le t\le b} |X_N(t) - X_M(t)| > 0\right] \le P[\tau_N \le b] = P\left[\sup_{a\le t\le b} |X_N(t)| > N\right]$$

Next we show that

$$\lim_{N\to\infty} P\left[\sup_{a\le t\le b} |X_N(t)| > N\right] = 0.$$

For this aim, consider the function $\phi(x) = \frac{1}{1+x^2}$. Applying Theorem 2.19 with $H(x) = x^2$ to the Itô process $X_N(t)$, we see that

$$\begin{aligned} X_N(t)^2 &= X_N(a)^2 + \int_a^b \left[2G_N(s, X_N(s))\, X_N(s) + F_N(s, X_N(s))^2\right] ds \\ &\quad +2\int_a^b F_N(s, X_N(s))\, X_N(s)\, dB(s) \end{aligned} \tag{3.15}$$

If we multiplying by $\phi[X_N(a)]$ and taking expectation in both sides of (3.15), we have

$$\begin{aligned} \mathbb{E}\Big[\phi[X_N(a)]\left[X_N(t)^2 - X_N(a)^2\right]\Big] &= \mathbb{E}\left[\phi[X_N(a)]\int_a^t \left[2G_N(s, X_N(s))\, X_N(s) + F_N(s, X_N(s))^2\right] ds\right] \\ &\quad +2\mathbb{E}\left[\phi[X_N(a)]\int_a^t F_N(s, X_N(s))\, X_N(s)\, dB(s)\right] \\ &= \mathbb{E}\left[\phi[X_N(a)]\int_a^t \left[2G_N(s, X_N(s))\, X_N(s) + G_N(s, X_s(t))^2\right] ds\right] \end{aligned}$$

because of $\phi[X_N(a)]$ is bounded and $F_N(t, X_N(t))\, X_N(t) \in L^2_{ad}([a,b]\times\Omega)$. Then,

$$\begin{aligned} \mathbb{E}\Big[\phi[X_N(a)]\left[X_N(t)^2 - X_N(a)^2\right]\Big] &= \mathbb{E}\Big[\phi[X_N(a)]\int_a^t \left[2G_N(s, X_N(s))\, X_N(s) + F_N(s, X_N(s))^2\right] ds\Big] \\ &\le 2\mathbb{E}\Big[\phi[X_N(a)]\int_a^t \left[G_N(s, X_N(s))\, X_N(s) + F_N(s, X_N(s))^2\right] ds\Big] \\ &\le 2\mathbb{E}\Big[\phi[X_N(a)]\int_a^t K^2\left[1 + X_N(s)^2\right] ds\Big] \\ &= 2K^2\mathbb{E}\Big[\phi[X_N(a)]\Big]t + 2K^2\mathbb{E}\Big[\phi[X_N(a)]\int_a^t X_N(s)^2\, ds\Big] \\ &= 2K^2\mathbb{E}\Big[\phi[X_N(a)]\Big]t + 2K^2\mathbb{E}\Big[\phi[X_N(a)]\int_a^t X_N(a)^2\, ds\Big] \\ &\quad +2K^2\mathbb{E}\Big[\phi[X_N(a)]\int_a^t \left[X_N(s)^2 - X_N(a)^2\right] ds\Big] \\ &= 2K^2\mathbb{E}\Big[\phi[X_N(a)]\Big]t + 2K^2\int_a^t \mathbb{E}\Big[\phi[X_N(a)]\, X_N(a)^2\Big] ds \\ &\quad +2K^2\int_a^t \mathbb{E}\Big[\phi[X_N(a)]\left[X_N(s)^2 - X_N(a)^2\right]\Big] ds \end{aligned}$$

By Gronwall-Bellman inequality (3.11) and $\phi[X_N(a)]\, X_N(a)^2 < 1$,

$$\mathbb{E}\Big[\phi[X_N(a)]\left[X_N(t)^2 - X_N(a)^2\right]\Big] \le 2K^2\left(1 + \mathbb{E}\Big[\phi[X_N(a)]\Big]\right)\left(t + 2K^2\int_a^t s e^{2K^2(t-s)} ds\right),$$

which implies that $\mathbb{E}\left[\phi\left[X_N(a)\right]X_N(t)^2\right] \leq H(K,t,X(a))$. Thus,

$$\mathbb{E}\left[\phi\left[X_N(a)\right]\sup_{a\leq t\leq b}X_N(t)^2\right] \leq H(K,t,X(a)).$$

Next, for $\delta > 0$,

$$\begin{aligned}
P\left[\sup_{a\leq t\leq b}|X_N(t)| > N\right] &= P\left[\phi\left[X(a)\right]\sup_{a\leq t\leq b}X_N(t)^2 > \phi\left[X(a)\right]N^2\right] \\
&= P\left[\phi\left[X(a)\right]\sup_{a\leq t\leq b}X_N(t)^2 > \phi\left[X(a)\right]N^2, \phi\left[X(a)\right] > \delta\right] \\
&\quad + P\left[\phi\left[X(a)\right]\sup_{a\leq t\leq b}X_N(t)^2 > \phi\left[X(a)\right]N^2, \phi\left[X(a)\right] \leq \delta\right] \\
&\leq P\left[\phi\left[X(a)\right]\sup_{a\leq t\leq b}X_N(t)^2 > \delta N^2\right] + P\left[\phi\left[X(a)\right]\leq\delta\right] \\
&\leq P\left[\phi\left[X(a)\right]\leq\delta\right] + \frac{H(K,t,X(a))}{\delta N^2}
\end{aligned}$$

Now,

$$\lim_{N\to\infty}P\left[\sup_{a\leq t\leq b}|X_N(t)| > N\right] \leq P\left[\phi\left[X(a)\right]\leq\delta\right] \tag{3.16}$$

Since $\delta > 0$ and $P\left[\phi\left[X(a)\right] = 0\right] = 0$, our aim has been proved. This fact implies that $X_N(t)$ converges uniformly with probability 1 to some limit $X(t)$ as $N\to\infty$, which satisfies the equality $X(t) = X_N(t)$ for all $t\leq\tau_N$.

Now we prove the uniqueness. Let $X_1(t)$ and $X_2(t)$ be two continuous solutions of (3.1) satisfying the initial condition $X_1(a) = X_2(a) = Z$. Define

$$\phi(t) = \begin{cases} 1 & \text{if} \quad \sup_{a\leq s\leq t}|X_1(s)| \leq N \text{ and } \sup_{a\leq s\leq t}|X_2(s)| \leq N \\ 0 & \text{otherwise} \end{cases}$$

Using condition (3.8),

$$\begin{aligned}
\mathbb{E}\left[X_1(t) - X_2(t)\right]^2\phi(t) &\leq 2\mathbb{E}\left[\phi(t)\left\{\int_a^t\left[G(s,X_1(s)) - G(s,X_2(s))\right]ds\right\}^2\right] \\
&\quad + 2\mathbb{E}\left[\phi(t)\left\{\int_a^t\left[F(s,X_1(s)) - F(s,X_2(s))\right]dB(s)\right\}^2\right] \\
&\leq 2t\mathbb{E}\left[\int_a^t\phi(s)\left[G(s,X_1(s)) - G(s,X_2(s))\right]^2 ds\right] \\
&\quad + 2\mathbb{E}\left[\int_a^t\phi(s)\left[F(s,X_1(s)) - F(s,X_2(s))\right]^2 ds\right] \\
&\leq (2b+2)L_N^2\int_a^t\mathbb{E}\left[\phi(s)\left[X_1(s) - X_2(s)\right]^2\right]ds.
\end{aligned}$$

By the Gronwall-Bellman inequality (3.11), we hold that

$$\mathbb{E}\left[\phi(s)\left[X_1(s) - X_2(s)\right]^2\right] = 0,$$

implying that

$$P\left[X_1(t)\neq X_2(t)\right]\leq P\left[\sup_{a\leq s\leq b}|X_1(s)|>N\right]+P\left[\sup_{a\leq s\leq b}|X_2(s)|>N\right].$$

As X_1 and X_2 are continuous, then X_1 and X_2 are bounded. This implies that the right side of the last inequality tend to zero as $N\to\infty$ and $P\left[X_1(t)=X_2(t)\right]=1$, as we want to show. □

Finally, we establish an estimate for the moments of solution of (3.12), following [8, p.48].

Theorem 3.6. *Assume that the coefficients of (3.1) and the initial condition Z satisfy the conditions of Theorem 3.4 and $X(t)$ is a solution of the SDE (3.1). If $\mathbb{E}\left[Z^{2m}\right]<\infty$, then*

$$\mathbb{E}\left[X(t)^{2m}\right]\leq\mathbb{E}\left[1+Z^{2m}\right]e^{2m(2m+1)K(t-a)}. \tag{3.17}$$

Moreover, if $G(t,x)\leq K_1\left(1+x^2\right)$ and $\mathbb{E}\left[Z^{4m}\right]<\infty$, then there exists a constant $L>0$ depending of K,K_1,m,a and b such that

$$\mathbb{E}\left[\left[X(t)-Z\right]^{2m}\right]\leq\tilde{K}\left[\mathbb{E}\left[Z^{4m}\right]+1\right](t-a)^m e^{2m(2m+1)K(t-a)}. \tag{3.18}$$

Proof. Let consider $F_N(t,x)$, $G_N(t,x)$ and Z_N as in Theorem 3.4. If $X_N(t)$ is a strong solution of the SDE

$$dX_N(t)=F_N(t,X_N(t))\,dB(t)+G_N(t,X_N(t))\,dt$$

with initial condition Z_N, then $X_N(t)$ is bounded, because $F_N(t,x)$ and $G_N(t,x)$ are bounded. Moreover,

$$|X_N(t)|\leq|Z_N|+H_1(N)\,B(t)+H_2(N)\,t$$

with $H_1(N)=K\sqrt{1+N^2}+2K_NN$ and $H_2(N)=K\frac{1+N^2}{N}+2K_NN$. Using the inequality $(a+b)^{2m}\leq 2^{2m-1}\left(a^{2m}+b^{2m}\right)$,

$$X_N(t)^{2m}\leq 2^{4m-2}\left(Z_N^{2m}+G(N)^{2m}t^{2m}\right)+2^{2m-1}F(N)^{2m}B(t)^{2m}$$

As $\mathbb{E}\left[B(t)^{2m}\right]<\infty$ and $\mathbb{E}\left[X_N(0)^{2m}\right]<\infty$, then $\mathbb{E}\left[X_N(t)^{2m}\right]<\infty$.

On the other hand, applying the Itô formula (Theorem 2.17) with $H(x)=x^{2m}$ to $X_N(t)$, we get

$$\begin{aligned}X_N(t)^{2m} &= Z_N^{2m}+\int_a^t\Big[2mX_N(s)^{2m-1}G_N(s,X_N(s))\\&\quad+m(2m-1)X_N(s)^{2m-2}F_N^2(s,X_N(s))\Big]ds\\&\quad+2m\int_a^t X_N(s)^{2m-1}F_N(s,X_N(s))\,dB(s).\end{aligned}$$

Observe that the function $X_N(s)^{2m-1} F_N(s, X_N(s)) \in L^2_{ad}([a,b]\times\Omega)$. Thus,

$$\mathbb{E}\left[\int_a^t X_N(s)^{2m-1} F_N(s, X_N(s))\, dB(s)\right] = 0.$$

By the last assertion,

$$\begin{aligned}
\mathbb{E}\left[X_N(t)^{2m}\right] &= \mathbb{E}\left[Z_N^{2m}\right] + \int_a^t \mathbb{E}\Big[2mX_N(s)^{2m-1} G_N(s, X_N(s)) \\
&\quad + m(2m-1)X_N(s)^{2m-2} F_N^2(s, X_N(s))\Big]\, ds \\
&\le \mathbb{E}\left[Z^{2m}\right] + (2m+1)mK\int_a^t \mathbb{E}\left[\left(1+X_N^2(s)\right)X_N(s)^{2m-2}\right] ds \\
&\le \mathbb{E}\left[Z^{2m}\right] + (2m+1)mK\int_a^t \mathbb{E}\left[1+2X_N^{2m}(s)\right] ds \\
&= \mathbb{E}\left[Z^{2m}\right] + (2m+1)mK(t-a) \\
&\quad +2(2m+1)mK\int_a^t \mathbb{E}\left[X_N^{2m}(s)\right] ds \\
&\le \mathbb{E}\left[Z^{2m}\right] + (2m+1)mK(t-a) \\
&\quad +2(2m+1)mk\int_a^t e^{2(2m+1)mK(t-s)}\left[\mathbb{E}\left[Z^{2m}\right]\right. \\
&\quad \left.+(2m+1)mK(s-a)\right] ds \\
&= \left[\mathbb{E}\left[Z^{2m}\right]+\frac{1}{2}\right]e^{2m(2m+1)K(t-a)} - \frac{1}{2} \\
&\le \left[\mathbb{E}\left[Z^{2m}\right]+1\right]e^{2m(2m+1)K(t-a)}
\end{aligned}$$

Finally, taking limit as $N\to\infty$,

$$\mathbb{E}\left[X(t)^{2m}\right] \le \left[\mathbb{E}\left[Z^{2m}\right]+1\right]e^{2m(2m+1)K(t-a)}.$$

For the second part of the theorem, let observe that

$$\begin{aligned}
\mathbb{E}\left[|X(t)-X(a)|^{2m}\right] &= \mathbb{E}\left[\left|\int_a^t G(s,X(s))\,ds + \int_a^t F(s,X(s))dB(s)\right|^{2m}\right] \\
&\le 2^{2m-1}\mathbb{E}\left[\left|\int_a^t G(s,X(s))\,ds\right|^{2m}\right] \\
&\quad +2^{2m-1}\mathbb{E}\left[\left|\int_a^t F(s,X(s))dB(s)\right|^{2m}\right] \\
&\le 2^{2m-1}(t-a)^{2m-1}\int_a^t \mathbb{E}\left[G(s,X(s))^{2m}\right] ds \\
&\quad +2^{2m-1}\left[m(2m-1)\right]^{m-1}(t-a)^{m-1}\int_a^t \mathbb{E}\left[F(s,X(s))^{2m}\right] ds \\
&\le 2^{2m-1}(b-a)^m(t-a)^{m-1}\int_a^t \mathbb{E}\left[G(s,X(s))^{2m}\right] ds \\
&\quad +2^{2m-1}\left[m(2m-1)\right]^{m-1}(t-a)^{m-1}\int_a^t \mathbb{E}\left[F(s,X(s))^{2m}\right] ds
\end{aligned}$$

Thus, there exists a constant $\bar{K}$ such that

$$\mathbb{E}\left[|X(t)-X(a)|^{2m}\right] \leq \bar{K}(t-a)^{m-1}\int_a^t \left[\mathbb{E}\left[G(s,X(s))^{2m}+F(s,X(s))^{2m}\right]ds\right]$$

As $F^2(t,x)\leq K\left(1+x^2\right)$ and $G(t,x)\leq K_1\left(1+x^2\right)$, there exists a constant $\mathbf{K}>0$ such that $F^2(t,x)+G^2(t,x)\leq \mathbf{K}\left(1+x^4\right)$. By this,

$$\begin{aligned}
\mathbb{E}\left[|X(t)-X(a)|^{2m}\right] &\leq \bar{K}(t-a)^{m-1}\int_a^t \left[\mathbb{E}\left[G(s,X(s))^{2m}+F(s,X(s))^{2m}\right]ds\right]\\
&\leq \bar{K}(t-a)^{m-1}\int_a^t \left[2\mathbf{K}^m\mathbb{E}\left[\left[1+X(s)^4\right]^m\right]ds\right]\\
&\leq 2^m\bar{K}\mathbf{K}^m(t-a)^{m-1}\int_a^t \left[1+\mathbb{E}\left[X(s)^{4m}\right]\right]ds\\
&\leq 2^m\bar{K}\mathbf{K}^m(t-a)^m\\
&\quad +2^m\bar{K}\mathbf{K}^m(t-a)^{m-1}\left[\mathbb{E}\left[Z^{4m}\right]+1\right]\int_a^t e^{C(s-a)}ds\\
&= 2^m\bar{K}\mathbf{K}^m(t-a)^m\\
&\quad +2^m\bar{K}\mathbf{K}^m(t-a)^m\left[\mathbb{E}\left[Z^{4m}\right]+1\right]\frac{e^{C(t-a)}-1}{C(t-a)}\\
&\leq 2^m\bar{K}\mathbf{K}^m(t-a)^m\\
&\quad +2^m\bar{K}\mathbf{K}^m(t-a)^m\left[\mathbb{E}\left[Z^{4m}\right]+1\right]e^{C(t-a)}\\
&\leq 2^{m+1}\bar{K}\mathbf{K}^m(t-a)^m\left[1+\mathbb{E}\left[Z^{4m}\right]\right]e^{C(t-a)}
\end{aligned}$$

with $C=4m(4m+1)K$. And finally, $\tilde{K}=2^{m+1}\bar{K}\mathbf{K}^m$. □

Remark 5. In the case that the coefficients $F(t,x)$ and $G(t,x)$ satisfies the condition (3.4):

$$F(t,x)^2+G(t,x)^2\leq K\left(1+x^2\right),$$

then there exists a constant $\tilde{K}$ depending of K, m, a and b for which

$$\mathbb{E}\left[[X(t)-Z]^{2m}\right]\leq \tilde{K}\left[\mathbb{E}\left[Z^{2m}\right]+1\right](t-a)^m e^{2m(2m+1)K(t-a)}. \tag{3.19}$$

3.2 Global solutions for SDE and Feller Criteria

Let $F(t,x), G(t,x)$ be functions defined on $[t_0,\infty)\times\mathbb{R}$ with $-\infty<t_0<\infty$. If the assumptions of the existence and uniqueness Theorem 3.4 hold on every finite subinterval $[t_0,b]\subset[t_0,\infty)$, then the SDE

$$dX(t)=F(t,X(t))\,dB(t)+G(t,X(t))\,dt$$

has a unique solution $X(t)$ defined on $[t_0,\infty)$. Such a solution is called a *global solution*.

Corollary 3.7. *Consider the autonomous stochastic differential equation*

$$dX(t) = F(X(t))\, dB(t) + G(X(t))\, dt. \tag{3.20}$$

For every initial value Z, independent of the Brownian motion $B(t) - B(t_0), t \in [t_0, \infty)$, the SDE (3.20) has a unique continuous global solution $X(t), t \in [t_0, \infty)$ such $X(t_0) = Z$ if the following Lipschitz condition is satisfied: there exists a positive constant K such that,

$$|F(x) - F(y)| + |G(x) - G(y)| \leq K|x - y| \ \forall\ x, y \in \mathbb{R}.$$

The growth condition of F and G follows from this global Lipschitz condition (only we fix $y = y_0$).

In the case the global Lipschitz condition is not satisfied, then the solution $X(t), t \in [t_0, \infty)$ can exhibit a phenomena known as explosion.

Definition 3.8. A random time $\tau \in [t_0, \infty)$ is a *explosion* if $P[X(\tau) = \infty] = 1$ or $P[X(\tau) = -\infty] = 1$. In this case, $X(t) = \infty$ or $X(t) = -\infty$ for all $t \geq \tau$, depending on the case.

There exists a result about the existence and uniqueness of solutions for the SDE (3.20), up to a time-explosion, in the case of the coefficients are continuous functions.

Theorem 3.9. *Let consider the autonomous SDE*

$$dX(t) = F(X(t))\, dB(t) + G(X(t))\, dt$$

with initial condition Z. If the coefficient $F(x), G(x)$ are continuously differentiable functions on $\mathbb{R}$, then there exists a global solution of the SDE until an explosion τ in the interval (t_0, τ) with $t_0 < \tau \leq \infty$.

For a proof of this theorem, see [16, Section 3.3 p. 54].

The Feller Criterion for Explosions in SDE

The so-called Feller test is a criterion to determine if a explosion occurs in finite or infinite time. For the statement of this test, let suppose that $X(t), t \in [t_0, \infty)$ is a solution of the autonomous SDE

$$dX(t) = F(X(t))\, dB(t) + G(X(t))\, dt \tag{3.21}$$

with initial condition $X(t_0) = x \in (a, b) \subseteq \mathbb{R}$, and the coefficients $F(x), G(x)$ satisfy the assumptions of the existence and uniqueness theorem 3.4 on $[t_0, \infty) \times \mathbb{R}$.

Let introduce two conditions on the coefficients of the SDE (3.21).

- non-degeneracy (ND): $F^2(x) > 0$ for all $x \in \mathbb{R}$.

- local integrability (LI): for all $x \in \mathbb{R}$ there exists an $\epsilon > 0$ such that

$$\int_{x-\epsilon}^{x+\epsilon} \frac{1+|G(y)|}{F^2(y)} dy < \infty.$$

Under these assumptions, for a fixed number $c \in \mathbb{R}$ define the *scale function*

$$S(x) := \int_c^x \exp\left[-2\int_c^y \frac{G(z)}{F^2(z)}dz\right] dy;\;\; x \in \mathbb{R}.$$

The derivative S' is continuous and strictly positive. Thus, S'' also exists and satisfies that

$$S''(x) = -2\frac{G(x)}{F^2(x)} S'(x).$$

Remark 6. The value of c does not determine $S(\pm\infty)$ is finite or not.

Let

$$T_{a,b} := \inf\{t \geq t_0;\;\; X(t) \notin (a,b)\};\;\; -\infty \leq a < b \leq \infty$$

be the *exit time from* (a,b), and $M(x)$ be the solution of the differential equation

$$\begin{aligned} \frac{1}{2}F^2(x) M''(x) + G(x) M'(x) &= -1,\;\; a < x < b, \\ M(a) = M(b) &= 0. \end{aligned} \tag{3.22}$$

Such a solution $M(x)$ can be expressed, in terms of the scale function, as follows:

$$M(x) = \frac{S(x)-S(a)}{S(b)-S(a)} \int_a^b \frac{S(b)-S(y)}{S'(y)F^2(y)}dy - \int_a^x \frac{S(x)-S(y)}{S'(y)F^2(y)}dy.$$

If we apply the Itô formula to the process $M(X(t)), t \geq t_0$, we obtain

$$M(X(t\wedge T_{a,b})) = M(x) - [t \wedge T_{a,b} - t_0] + \int_{t_0}^{t\wedge T_{a,b}} M'(X(s)) F(X(s)) dB(s).$$

Taking expectation, we see that

$$\mathbb{E}[t\wedge T_{a,b}] = M(x) + t_0 - \mathbb{E}[M(X(t\wedge T_{a,b}))] \leq M(x) + t_0 < \infty,$$

and letting $t \to \infty$ we obtain $\mathbb{E}[T_{a,b}] \leq M(x) < \infty$. In other words, $X(t), t \in [t_0, \infty)$ exits from every bounded subinterval of $\mathbb{R}$ in finite expected time. Moreover, $\mathbb{E}[T_{a,b}] = M(x)$.

In the same way, if we apply the Itô formula to the process $S(X(t)), t \geq t_0$, we obtain that $S(x) = \mathbb{E}[S(X(t\wedge T_{a,b}))]$ and

$$S(x) = \mathbb{E}[S(X(T_{a,b}))] = S(a) P[X(T_{a,b}) = a] + S(b) P[X(T_{a,b}) = b].$$

The two probabilities in the last equation add up to one, and thus

$$P[X(T_{a,b}) = a] = \frac{S(b)-S(x)}{S(b)-S(a)};\;\; P[X(T_{a,b}) = b] = \frac{S(x)-S(a)}{S(b)-S(a)}.$$

Proposition 3.10. *Assume that conditions ND and LI are satisfied for the functions* $F(x)$ *and* $G(x)$ *and let* $X(t), t \geq t_0$ *be a global solution of the SDE*

$$dX(t) = F(X(t))\,dB(t) + G(X(t))\,dt \tag{3.23}$$

with initial condition $X(t_0) = x \in (a, b)$*. Let* S *be the scale function,* $S(\pm\infty) := \lim_{x \to \pm\infty} S(x)$ *and* τ *the explosion-time of* $X(t), t \geq t_0$*. We distinguish four cases:*

(a) If $S(-\infty) = -\infty$ *and* $S(\infty) = \infty$*, then*

$$P[\tau = \infty] = P\left[\sup_{t_0 \leq t < \infty} X(t) = \infty\right] = P\left[\inf_{t_0 \leq t < \infty} X(t) = -\infty\right] = 1.$$

In particular, the process $X(t), t \in [t_0, \infty)$ *is recurrent: for every* $y \in \mathbb{R}$*, we have*

$$P[X(t) = y;\ \textit{some } t_0 \leq t < \infty] = 1.$$

(b) If $S(-\infty) > -\infty$ *and* $S(\infty) = \infty$*, then*

$$P\left[\sup_{t_0 \leq t < \tau} X(t) < \infty\right] = P\left[\lim_{t \to \tau^+} X(t) = -\infty\right] = 1.$$

(c) If $S(-\infty) = -\infty$ *and* $S(\infty) < \infty$*, then*

$$P\left[\inf_{t_0 \leq t < \tau} X(t) > -\infty\right] = P\left[\lim_{t \to \tau^+} X(t) = \infty\right] = 1.$$

(d) If $S(-\infty) > -\infty$ *and* $S(\infty) < \infty$*, then*

$$P\left[\lim_{t \to \tau^+} X(t) = -\infty\right] = 1 - P\left[\lim_{t \to \tau^+} X(t) = \infty\right] = \frac{S(\infty) - S(x)}{S(\infty) - S(-\infty)}.$$

Remark 7. Unless cases (b), (c) and (d) does not make no claim concerning the finiteness of τ, it is possible to find examples in each one such that $P[\tau = \infty] = 1$. For cases (b) and (c), the Brownian motion with drift $X(t) = Z + \mu t + \sigma B(t), t \geq t_0$ is solution of the SDE

$$dX(t) = \mu dt + \sigma dB(t),\ X(t_0) = Z,$$

and it holds that $P[X(\tau) = \infty] = 1$ if $\mu > 0$ and $P[X(\tau) = -\infty] = 1$ if $\mu > 0$. In contrast, if $F(x) = \mu$ and $G(x) = sgn(x)$, then the SDE

$$dX(t) = sgn(X(t))\,dt + \sigma dB(t),\ X(t_0) = Z$$

has a unique non-explosive strong solution $X(t), t \geq t_0$ (see [12, p.342, Remark 5.18]).

The proof of this proposition is given in [12, Proposition 5.22 p.345]. In order to find necessary and sufficient conditions to guarantee that $P[\tau=\infty]=1$, we will use a result from ordinary differential equations. For that, define recursively the sequence $\{u_n(x)\}_{n=0}^{\infty}$ of real-valued functions on $\mathbb{R}$ by setting $u_0=1$ and

$$u_n(x)=2\int_c^x S'(y)\left[\int_c^y \frac{u_{n-1}(z)}{S'(z)F^2(z)}dz\right]dy$$

where, as before, c is a fixed number in $\mathbb{R}$. In particular, we set for $x\in\mathbb{R}$:

$$v(x):=u_1(x)=2\int_c^x \frac{S(x)-S(y)}{S'(y)F^2(y)}dy.$$

Lemma 3.11. *Assume that conditions ND and LI are satisfied for the functions $F(x)$ and $G(x)$. The series*

$$u(x):=\sum_{n=0}^{\infty}u_n(x);\;\; x\in\mathbb{R}$$

converges uniformly on compact subsets of $\mathbb{R}$ and $u(x)$ is a differentiable function with absolutely continuous derivative on $\mathbb{R}$. Furthermore, u is strictly increasing for $x>c$, strictly decreasing for $x<c$, satisfies

$$\begin{aligned}\frac{1}{2}F^2(x)u''(x)+G(x)u'(x) &= u(x);\;\; a.s.\; x\in\mathbb{R}\\ u(c)=1, &\qquad u'(c)=0,\end{aligned}$$

and the inequalities

$$1+v(x)\le u(x)\le e^{v(x)};\;\; \forall x\in\mathbb{R}.$$

The proof of this lemma is given in [12, Lemma 5.26, p.347]. Now, we present the Feller test for explosions.

Theorem 3.12. *Let $F(x),G(x)$ defined over $\mathbb{R}$ that satisfies conditions (ND) and (LI) and let $X(t),t\ge t_0$ be a global solution of the SDE*

$$dX(t)=F(X(t))dB(t)+G(X(t))dt \tag{3.24}$$

with initial condition $X(t_0)=x\in\mathbb{R}$. Then, $P[\tau=\infty]=1$ or $P[\tau=\infty]<1$, according to whether $v(-\infty)=v(\infty)=\infty$ or not.

Proof. Let $Z_n(t):=u(X(t\wedge\tau_n\wedge T_{-n,n}))$ where

$$\tau_n:=\inf\{t\ge t_0;\int_{t_0}^t F^2(X(s))ds\ge n\}.$$

By the Itô formula,

$$Z_n(t)=Z_n(t_0)+\int_{t_0}^{t\wedge\tau_n\wedge T_{-n,n}}u(X(s))ds+\int_{t_0}^{t\wedge\tau_n\wedge T_{-n,n}}u'(X(s))F(X(s))dB(s).$$

Next, $M_n(t) := e^{-(t\wedge\tau_n\wedge T_{-n,n})} Z_n(t)$ has the representation

$$M_n(t) = M_n(t_0) + \int_{t_0}^{t\wedge\tau_n\wedge T_{-n,n}} e^{-s} u'(X(s)) F(X(s))\, dB(s).$$

Let define $M(t) := \lim_{n\to\infty} M_n(t) = e^{-(t\wedge\tau)} u(X(t\wedge\tau))$. Thus, the process

$$M(t), t \geq t_0$$

is a non-negative super martingale and this implies that

$$M(\infty) := \lim_{t\to\infty} M(t)$$

exists and it is finite, almost surely.

- Suppose that $v(-\infty) = v(\infty) = \infty$. Then, the inequality (3.11), $u(-\infty) = u(\infty) = \infty$. On the other hand, $M(\infty) = \infty$ a.s. on the event $[\tau < \infty]$. Thus implies that $P[\tau < \infty] = 0$.
- Now suppose that $v(\infty) < \infty$. By the inequality (3.11), $u(\infty) < \infty$. Without lost of generality, let assume that $c < x$ and define $T_c := \inf\{t \geq t_0; X(t) = c\}$. The process

  $$M(t\wedge T_c) = e^{-(t\wedge\tau\wedge T_c)} u(X(t\wedge\tau\wedge T_c)), \quad t_0 \leq t < \infty,$$

  is a bounded martingale, which therefore converges almost surely as $t \to \infty$. Thus,

  $$u(x) = \mathbb{E}\left[e^{-(\tau\wedge T_c)} u(X(\tau\wedge T_c))\right] = u(\infty)\,\mathbb{E}\left[e^{-\tau} 1_{[\tau<T_c]}\right] + u(c)\,\mathbb{E}\left[e^{-T_c} 1_{[T_c<\tau]}\right].$$

  Now, if $P[\tau = \infty] = 1$, it follows that $u(x) = u(c)\,\mathbb{E}\left[e^{-T_c}\right] \leq u(c)$, which contradict the hypothesis of strictly increasing for $x > c$. Then, $P[\tau = \infty] < 1$.

□

3.3 Properties of the solution of SDE

One of the principal families of stochastic processes are the Markov processes.

Definition 3.13. A stochastic process $X(t), t \in [a,b]$ defined on a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{a\leq t\leq b}, P)$ taking values in $\mathbb{R}$ is said to be a *Markov process* if it satisfies the so-called Markov property holds:

$$P[X(t) \in A | \mathcal{F}_s] = P[X(t) \in A | X(s)] \quad \forall s \leq t,\ A \in \mathcal{B}(\mathbb{R}).$$

If in addition, there exists a function $P(s,x,t,A)$ satisfying the next conditions

- $P(s,x,t,A)$ is a Borel function of x,

- $P(s,x,t,A)$ is measurable with respect to A for fixed s,t,x,
- $\int P(s,x,u,dy)\, P(u,y,t,A) = P(s,x,t,A)$ for all x and $a \le s < u < t \le b$,

such that

$$P[X(t) \in A|\mathcal{F}_s] = P(s, X(s), t, A),$$

for all $x \in \mathbb{R}$, $a \le s < t \le b$ and $A \in \mathcal{B}(\mathbb{R})$, then the function $P(s,x,t,A)$ is called the *transition probability of the Markov process* $X(t), t \in [a,b]$.

Theorem 3.14. *Let $X(t), t \in [a,b]$ be a Markov process and $P(s,x,t,A)$ its transition probability function. If $H(t,x)$ is strictly monotone in x for all $t \in [a,b]$, then $H(t, X(t)), t \in [a,b]$ is also a Markov process and its transition probability*

$$\tilde{P}(s,x,t,A)$$

satisfies that

$$\tilde{P}(s,x,t,A) = P\left(s, H^{-1}(s,x), t, H^{-1}(t,A)\right)$$

where $H^{-1}(s,x)$ is the inverse of $H(t,x)$ in x and $H^{-1}(t,A) = \{y \in \mathbb{R} \mid H(t,y) \in A\}$.

Proof. First, let show that $Y(t) = H(t, X(t)), t \in [a,b]$ is a Markov process. For that, let $\mathcal{G}_t := \sigma(Y(s), a \le s \le t)$. Thus,

$$\begin{aligned} P[Y(t) \in A|\mathcal{G}_s] &= P[H(t, X(t)) \in A|\mathcal{G}_s] \\ &= P\left[H^{-1}(t, H(t, X(t))) \in H^{-1}(t,A)\,|\mathcal{F}_s\right] \\ &= P\left[X(t) \in H^{-1}(t,A)\,|\mathcal{F}_s\right] \\ &= P\left[X(t) \in H^{-1}(t,A)\,|X(s)\right] \\ &= P\left[H^{-1}(t, H(t, X(t))) \in H^{-1}(t,A)\,|X(s)\right] \\ &= P[H(t, X(t)) \in A|H(s, X(s))] \\ &= P[Y(t) \in A|Y(s)] \end{aligned}$$

Moreover, if $\tilde{P}(s,x,t,A)$ is the transition probability function of $Y(t) = H(t, X(t)), t \in [a,b]$, then

$$\begin{aligned} \tilde{P}(s, Y(s), t, A) &= P[Y(t) \in A|\mathcal{G}_s] \\ &= P\left[H^{-1}(t, Y(t)) \in H^{-1}(t,A)\,|\mathcal{F}_s\right] \\ &= P\left[X(t) \in H^{-1}(t,A)\,|\mathcal{F}_s\right] \\ &= P\left(s, X(s), t, H^{-1}(t,A)\right) \\ &= P\left(s, H^{-1}(s, Y(s)), t, H^{-1}(t,A)\right) \end{aligned}$$

as we want to show. □

In the case of stochastic differential equations, if $X(t), t \in [a,b]$, is a solution of the SDE

$$dX(t) = F(t, X(t))\, dB(t) + G(t, X(t))\, dt \tag{3.25}$$

with initial condition Z and coefficients F and G satisfying the assumptions of Theorem 3.4, then $X(t), t \in [a,b]$ belongs to the family of Markov process, as it is given in [8, Thm. 1, p.67].

Theorem 3.15. *The solution $X(t), t \in [a,b]$ of the SDE (3.25) is a Markov process, whose transition probability is defined by*

$$P(s,x,t,A) := P[X_{s,x}(t) \in A], \tag{3.26}$$

where $X_{s,x}(t), t \in [s,b]$ is a solution of the SDE

$$X_{s,x}(t) = x + \int_s^t F(u, X_{s,x}(u))\, dB(u) + \int_s^t G(u, X_{s,x}(u))\, du \tag{3.27}$$

on the interval $[t,b]$.

An important subfamily of Markov processes are the diffusion processes, which are defined below.

Definition 3.16. A Markov process $X(t), t \in [a,b]$ with transition probability $P(s,x,t,A)$ is said to be a *diffusion process* if the following properties hold:

1. for any $\epsilon > 0$ and $a \leq t \leq b$, $x \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{1}{h} \int_{|x-y|>\epsilon} P(t,x,t+h,dy) = 0$$

2. there exist functions $A(t,x)$ and $B(t,x)$ such that for all $\epsilon > 0$, $t \in T$ and $x \in \mathbb{R}$,

(a)

$$\lim_{h \to 0} \frac{1}{h} \int_{|x-y|\leq\epsilon} (y-x)\, P(t,x,t+h,dy) = A(t,x)$$

(b)

$$\lim_{h \to 0} \frac{1}{h} \int_{|x-y|\leq\epsilon} (y-x)^2\, P(t,x,t+h,dy) = B(t,x)$$

The functions $A(t,x)$ and $B(t,x)$ are called the coefficient of displacement(drift) and the coefficient of diffusion, respectively.

Remark 8. For $X(t), t \in [a,b]$ to be a diffusion it is sufficient that its transition probability satisfy the following assumptions:

1* for any $\delta > 0$,

$$\lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} |x-y|^{2+\delta} P(t,x,t+h,dy) = 0 \tag{3.28}$$

2* there exists functions $A(t,x)$ and $B(t,x)$ such that for all $t \in T$ and $x \in \mathbb{R}$,

(a)

$$\lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} (y-x)\, P(t,x,t+h,dy) = A(t,x)$$

(b)
$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}(y-x)^2 P(t,x,t+h,dy)=B(t,x)$$

In fact, if 1* holds, then

$$\begin{aligned}
\int_{|y-x|>\epsilon} P(t,x,t+h,dy) &= \int_{|y-x|>\epsilon}\frac{|y-x|^{2+\delta}}{|y-x|^{2+\delta}}P(t,x,t+h,dy)\\
&\leq \int_{|y-x|>\epsilon}\frac{|y-x|^{2+\delta}}{\epsilon^{2+\delta}}P(t,x,t+h,dy)\\
&\leq \frac{1}{\epsilon^{2+\delta}}\int_{|y-x|>\epsilon}|y-x|^{2+\delta}P(t,x,t+h,dy)\\
&\leq \frac{1}{\epsilon^{2+\delta}}\int_{\mathbb{R}}|y-x|^{2+\delta}P(t,x,t+h,dy).
\end{aligned}$$

Moreover, condition 2* implies that

$$\begin{aligned}
\left|\int_{|y-x|>\epsilon}(y-x)P(t,x,t+h,dy)\right| &\leq \int_{|y-x|>\epsilon}|y-x|P(t,x,t+h,dy)\\
&= \int_{|y-x|>\epsilon}\frac{|y-x|^{2+\delta}}{|y-x|^{1+\delta}}P(t,x,t+h,dy)\\
&\leq \int_{|y-x|>\epsilon}\frac{|y-x|^{2+\delta}}{\epsilon^{1+\delta}}P(t,x,t+h,dy)\\
&\leq \frac{1}{\epsilon^{1+\delta}}\int_{|y-x|>\epsilon}|x-y|^{2+\delta}P(t,x,t+h,dy)\\
&\leq \frac{1}{\epsilon^{1+\delta}}\int_{\mathbb{R}}|y-x|^{2+\delta}P(t,x,t+h,dy),
\end{aligned}$$

and also that

$$\begin{aligned}
\int_{|y-x|>\epsilon}(y-x)^2 P(t,x,t+h,dy) &= \int_{|y-x|>\epsilon}\frac{|y-x|^{2+\delta}}{|y-x|^{\delta}}P(t,x,t+h,dy)\\
&\leq \int_{|y-x|>\epsilon}\frac{|y-x|^{2+\delta}}{\epsilon^{\delta}}P(t,x,t+h,dy)\\
&\leq \frac{1}{\epsilon^{\delta}}\int_{|y-x|>\epsilon}|x-y|^{2+\delta}P(t,x,t+h,dy)\\
&\leq \frac{1}{\epsilon^{\delta}}\int_{\mathbb{R}}|y-x|^{2+\delta}P(t,x,t+h,dy).
\end{aligned}$$

Thus, if equation (3.28) is satisfied, then the definition of the diffusion is satisfied.

Theorem 3.17. *Let $H(t,x)$ be twice continuously differentiable and monotone in x and continuously differentiable in t and let $X(t), t\in[a,b]$ be a diffusion. Then the process $H(t,X(t)), t\in[a,b]$ is also a diffusion with the drift coefficient is*

$$\begin{aligned}
\tilde{A}(t,x) &= H_t\left(t,H^{-1}(t,x)\right)+A\left(t,H^{-1}(t,x)\right)H_x\left(t,H^{-1}(t,x)\right)\\
&\quad+\frac{1}{2}B\left(t,H^{-1}(t,x)\right)H_{xx}\left(t,H^{-1}(t,x)\right)
\end{aligned}$$

and the diffusion coefficient is

$$\tilde{B}(t,x) = B\left(t, H^{-1}(t,x)\right)\left[H_x\left(t, H^{-1}(t,x)\right)\right]^2 \tag{3.29}$$

Proof. By Theorem 3.14, $H(t, X(t))$ is a Markov process. Now, let show Remark 8 is satisfied. Let $u = H^{-1}(t,x)$ and $v = H^{-1}(t+h,y)$. Then,

$$\begin{aligned}\frac{1}{h}\int_{|y-x|>\epsilon} \tilde{P}(t,x,t+h,dy) &= \frac{1}{h}\int_{|y-x|>\epsilon} P\left(t, H^{-1}(t,x), t+h, H^{-1}(t+h,dy)\right) \\ &= \frac{1}{h}\int_{|H(t+h,v)-H(t,u)|>\epsilon} P(t,u,t+h,dv) = \frac{o(h)}{h} \to 0\end{aligned}$$

because $X(t)$ is a diffusion. Next, the expansion on Taylor series of $H(t,x)$ around $(t+h,v)$ is

$$\begin{aligned}H(t+h,v) &= H(t,u) + \frac{\partial}{\partial t}H(t,u)\,h + \frac{\partial}{\partial u}H(t,u)(v-u) \\ &\quad + \frac{1+o(\epsilon)}{2}\frac{\partial^2}{\partial u^2}H(t,u)(v-u)^2 + o(h)\end{aligned}$$

with $|H(t+h,v) - H(t,u)| \leq \epsilon$. Then,

$$\begin{aligned}\int_{|y-x|\leq\epsilon}(y-x)\tilde{P}(t,x,t+h,dy) &= \int_{|y-x|\leq\epsilon}(y-x)P\left(t,H^{-1}(t,x),t+h,H^{-1}(t+h,dy)\right) \\ &= \int_{|H(t+h,v)-H(t,u)|\leq\epsilon}(H(t+h,v)-H(t,u))P(t,u,t+h,dv) \\ &= \left[o(h) + h\frac{\partial}{\partial t}H(t,u)\right]\int_{|H(t+h,v)-H(t,u)|\leq\epsilon}P(t,u,t+h,dv) \\ &\quad + h\frac{\partial}{\partial u}H(t,u)\int_{|H(t+h,v)-H(t,u)|\leq\epsilon}(v-u)P(t,u,t+h,dv) \\ &\quad + h\frac{1+o(\epsilon)}{2}\frac{\partial^2}{\partial u^2}H(t,u)\int_{|H(t+h,v)-H(t,u)|\leq\epsilon}(v-u)^2P(t,u,t+h,dv)\end{aligned}$$

Taking limits in both sides of the last equation, we hold that

$$\begin{aligned}\tilde{A}(t,x) &= \lim_{h\to 0}\frac{1}{h}\int_{|y-x|\leq\epsilon}(y-x)\tilde{P}(t,x,t+h,dy) \\ &= \frac{\partial}{\partial t}H(t,u) + A(t,u)\frac{\partial}{\partial u}H(t,u) + \frac{1}{2}B(t,u)\frac{\partial^2}{\partial u^2}H(t,u) \\ &= \frac{\partial}{\partial t}H\left(t,H^{-1}(t,x)\right) + A\left(t,H^{-1}(t,x)\right)\frac{\partial}{\partial x}H\left(t,H^{-1}(t,x)\right) \\ &\quad + \frac{1}{2}B\left(t,H^{-1}(t,x)\right)\frac{\partial^2}{\partial x^2}H\left(t,H^{-1}(t,x)\right)\end{aligned}$$

Analogously,

$$\begin{aligned}
\tilde{B}(t,x) &= \lim_{h\to 0}\frac{1}{h}\int_{|y-x|\le\epsilon}(y-x)^2\,\tilde{P}(t,x,t+h,dy)\\
&= \lim_{h\to 0}\frac{1}{h}\int_{|y-x|\le\epsilon}(y-x)^2\,P\left(t,H^{-1}(t,x),t+h,H^{-1}(t+h,dy)\right)\\
&= \lim_{h\to 0}\frac{1}{h}\int_{|H(t+h,v)-H(t,u)|\le\epsilon}\left[H(t+h,v)-H(t,u)\right]^2 P(t,u,t+h,dv)\\
&= \lim_{h\to 0}\frac{1}{h}\int_{|H(t+h,v)-H(t,u)|\le\epsilon}\left[(1+o(\epsilon))\frac{\partial}{\partial u}H(t,u)(v-u)+o(h)\right]^2 P(t,u,t+h,dv)\\
&= \left[\frac{\partial}{\partial u}H(t,u)\right]^2\lim_{h\to 0}\frac{(1+o(\epsilon))^2}{h}\int_{|H(t+h,v)-H(t,u)|\le\epsilon}(v-u)^2\,P(t,u,t+h,dv)\\
&= B(t,u)\left[\frac{\partial}{\partial u}H(t,u)\right]^2 = B\left(t,H^{-1}(t,x)\right)\left[\frac{\partial}{\partial x}H\left(t,H^{-1}(t,x)\right)\right]^2
\end{aligned}$$

□

The assumptions of Remark 8 provides an easy way to show if a Markov process $X(t), t\in[a,b]$ is a diffusion process. The next theorem shows that a solution of a SDE, under certain conditions, is a diffusion process.

Theorem 3.18. *Let $F(t,x)$ and $G(t,x)$ be continuous in both arguments and assume that*

(a) for all $t\in[a,b]$ and each $N>0$, there exists $L_N>0$ such that

$$|F(t,x)-F(t,y)|+|G(t,x)-G(t,y)|\le L_N|x-y|$$

for $(x,y)\in[-N,N]\times[-N,N]$,

(b) for all $t\in[a,b]$, $x\in\mathbb{R}$, there exists $K>0$ such that

$$G(t,x)+xG(t,x)+|F(t,x)|^2\le K\left(1+x^2\right).$$

Then the solution $X(t), t\in[a,b]$ of (3.25) is a diffusion process with displacement coefficient $A(t,x)=G(t,x)$ and diffusion coefficient $B(t,x)=F^2(t,x)$.

Proof. Let $X_{t,x}(s)$ be as in Theorem 3.15. Then, by Theorem 3.6, there exists a constant K, independent of t and x such that

$$\begin{aligned}
\int(y-x)^4\,P(t,x,t+h,dy) &= \mathbb{E}\left[|X_{t,x}(t+h)-X_{t,x}(t)|^4\right]\\
&= \mathbb{E}\left[|X_{t,x}(t+h)-x|^4\right]\\
&\le Kh^2\left(1+x^8\right)
\end{aligned}$$

Thus,

$$\lim_{h\to 0}\frac{1}{h}\int(y-x)^4\,P(t,x,t+h,dy)\le\lim_{h\to 0}Kh\left(1+x^8\right)=0.$$

Next,

$$\begin{aligned}\frac{1}{h}\mathbb{E}\left[X_{t,x}(t+h)-x\right] &= \frac{1}{h}\int_t^{t+h}\mathbb{E}\left[G\left(u,X_{t,x}(u)\right)\right]du\\ &= \int_0^1\mathbb{E}\left[G\left(t+sh,X_{t,x}(t+sh)\right)\right]ds.\end{aligned}$$

Since $G(t+sh,X_{t,x}(t+sh))\to G(t,x)$ a.s., as $h\to 0$, then

$$G(t+sh,X_{t,x}(t+sh))\le K\left(1+X_{t,x}(t+sh)^2\right)$$

and $\int_0^1\mathbb{E}\left[\left(1+X_{t,x}(t+sh)^2\right)\right]ds<\infty$. Thus, the hypothesis of the dominated convergence theorem of Lebesgue is satisfied and

$$\lim_{h\to 0}\int_0^1\mathbb{E}\left[G\left(t+sh,X_{t,x}(t+sh)\right)\right]ds=G(t,x).$$

On other hand, the Itô formula 2.19 for $H(x)=x^2$ yields

$$\begin{aligned}\mathbb{E}\left[\left|X_{t,x}(t+h)-x\right|^2\right] &= \mathbb{E}\left[X_{t,x}(t+h)^2-2xX_{t,x}(t+h)+x^2\right]\\ &= \mathbb{E}\left[X_{t,x}(t+h)^2\right]-x^2-2x\left(\mathbb{E}\left[X_{t,x}(t+h)\right]-x\right)\\ &= \mathbb{E}\left[\int_t^{t+h}\left[2X_{t,x}(u)G\left(u,X_{t,x}(u)\right)+F^2\left(u,X_{t,x}(u)\right)\right]du\right]\\ &\quad +2\mathbb{E}\left[\int_t^{t+h}X_{t,x}(u)F\left(u,X_{t,x}(u)\right)dB(u)\right]\\ &\quad -2x\left[G(t,x)h+o(h)\right].\end{aligned}$$

In an analogous manner, we see that

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[\left|X_{t,x}(t+h)-x\right|^2\right] &= 2\lim_{h\to 0}\int_0^1\mathbb{E}\left[X_{t,x}(t+sh)G\left(t+sh,X_{t,x}(t+sh)\right)\right]ds\\ &\quad +\lim_{h\to 0}\int_0^1\mathbb{E}\left[F^2\left(t+sh,X_{t,x}(t+sh)\right)\right]ds-2xG(t,x)\\ &= 2xG(t,x)+F^2(t,x)-2xG(t,x)=F^2(t,x).\end{aligned}$$

□

3.4 Boundaries classification of a diffusion processes

Let $X(t), t\in[t_0,\infty)$ be a global solution up to an explosion-time τ of the autonomous SDE

$$dX(t)=F(X(t))\,dB(t)+G(X(t))\,dt, \tag{3.30}$$

with initial condition Z, and suppose the coefficients F and G satisfies the assumptions of theorem 3.18. Recall the scale function $S(x)$ and the exit time $T_{a,b}$ studied in Section 3.2.

In this section, we present a classification about which states can be attached by the process and if that states can be handle in finite time.

Attracting states. Let start the discussion about if a state can be attached or not.

Definition 3.19. Let $l \in \mathbb{R}$ and $b \in (l, \infty)$. We said

- l is an *attracting state* if $P\left[X\left(T_{l,b}\right) = l | Z = x\right] > 0$ for all $x \in (l, b)$.
- l is a *non-attracting state* if $P\left[X\left(T_{l,b}\right) = l | Z = x\right] = 0$ for all $x \in (l, b)$.

The next proposition characterizes the behaviour of the process $X\left(t\right), t \geq t_0$ near l.

Proposition 3.20. *(a) If* $\lim_{a \to l^+} \int_a^b S'\left(u\right) du < \infty$ *for some* $b > l$*, then* l *is an attracting state;*

(b) If $\lim_{a \to l^+} \int_a^b S'\left(u\right) du = \infty$ *for some* $b > l$*, then* l *is a non-attracting state.*

Proof. By equation (3.2), we hold that

$$P\left[X\left(T_{a,b}\right) = a | Z = x\right] = \frac{\int_x^b S'\left(u\right) du}{\int_a^b S'\left(u\right) du}$$

and also,

$$\begin{aligned} P\left[X\left(T_{l,b}\right) = l | Z = x\right] &= \lim_{a \to l^+} P\left[X\left(T_{a,b}\right) = a | Z = x\right] \\ &= \lim_{a \to l^+} \frac{\int_x^b S'\left(u\right) du}{\int_a^b S'\left(u\right) du} \\ &= \left[\int_x^b S'\left(u\right) du\right] \lim_{a \to l^+} \frac{1}{\int_a^b S'\left(u\right) du}. \end{aligned}$$

If $\lim_{a \to l^+} \int_a^b S'\left(u\right) du < \infty$, then $\int_x^b S'\left(u\right) du < \infty$ and thus, $P\left[X\left(T_{l,b}\right) = l | Z = x\right] > 0$. On the other hand, if $\lim_{a \to l^+} \int_a^b S'\left(u\right) du = \infty$, then $\lim_{a \to l^+} \frac{1}{\int_a^b S'(u) du} = 0$ and thus, $P\left[X\left(T_{l,b}\right) = l | Z = x\right] = 0$. □

The same discussion can be presented for a right state $r > x$.

Definition 3.21. Let $r \in \mathbb{R}$ and $a \in (-\infty, a)$. We said

- r is an *attracting state* if $P\left[X\left(T_{a,r}\right) = r | Z = x\right] > 0$ for all $x \in (a, r)$.
- r is a *non-attracting state* if $P\left[X\left(T_{a,r}\right) = r | Z = x\right] = 0$ for all $x \in (a, r)$.

Moreover, we hold the characterization oh the behaviour of the process $X\left(t\right), t \geq t_0$ near r.

Proposition 3.22. *(a) If* $\lim_{b\to r^-}\int_a^b S'(u)\,du < \infty$ *for some* $a<r$*, then* r *is an attracting state;*

(b) If $\lim_{b\to r^-}\int_a^b S'(u)\,du = \infty$ *for some* $a<r$*, then* r *is a non-attracting state.*

Attainable states. Now, let discuss about if a state can be reached in finite time or not.

Definition 3.23. Let $l\in\mathbb{R}$ and $b\in(l,\infty)$. We said

- l is an *attainable state* if for all $x\in(l,b)$, $\mathbb{E}\left[T_{l,b}\middle|Z=x\right]<\infty$.
- l is an *unattainable state* if for all $x\in(l,b)$, $\mathbb{E}\left[T_{l,b}\middle|Z=x\right]=\infty$.

The next proposition characterizes the reachability of the state l by the process $X(t), t\geq t_0$.

Proposition 3.24. *(a) If* l *is an attracting state and*

$$\lim_{a\to l^+}\int_a^x \frac{S(z)-S(a)}{S'(z)\,F^2(z)}dz<\infty, \tag{3.31}$$

for some $x>l$*, then* l *is an attainable state.*

(b) If l *is an attracting and*

$$\lim_{a\to l^+}\int_a^x \frac{S(z)-S(a)}{S'(z)\,F^2(z)}dz=\infty \tag{3.32}$$

for some $x>l$*, then* l *is an unattainable state.*

(c) If l *is a non-attracting state, then* l *is an unattainable state.*

Proof. Recall the function $M(x)$ that is solution of the ordinary differential equation (3.22). This functions satisfies the equality $M(x)=\mathbb{E}\left[T_{a,b}\middle|Z=x\right]$. By straightforward calculus, we hold that,

$$M(x)=2\frac{S(x)-S(a)}{S(b)-S(a)}\int_x^b\frac{S(b)-S(z)}{S'(z)\,F^2(z)}dz+2\frac{S(b)-S(x)}{S(b)-S(a)}\int_a^x\frac{S(z)-S(a)}{S'(z)\,F^2(z)}dz \tag{3.33}$$

Thus,

$$\begin{aligned}\mathbb{E}\left[T_{l,b}\middle|Z=x\right] &= \lim_{a\to l^+}\mathbb{E}\left[T_{a,b}\middle|Z=x\right]\\ &= 2\lim_{a\to l^+}\frac{S(x)-S(a)}{S(b)-S(a)}\int_x^b\frac{S(b)-S(z)}{S'(z)\,F^2(z)}dz\\ &\quad +2\lim_{a\to l^+}\frac{S(b)-S(x)}{S(b)-S(a)}\int_a^x\frac{S(z)-S(a)}{S'(z)\,F^2(z)}dz\\ &= 2\int_x^b\frac{S(b)-S(z)}{S'(z)\,F^2(z)}dz\lim_{a\to l^+}\frac{S(x)-S(a)}{S(b)-S(a)}\\ &\quad +2\lim_{a\to l^+}\frac{S(b)-S(x)}{S(b)-S(a)}\int_a^x\frac{S(z)-S(a)}{S'(z)\,F^2(z)}dz\end{aligned} \tag{3.34}$$

In case (a), the r.h.s. of (3.34) is finite. This implies that $\mathbb{E}\left[T_{l,b}\middle|Z=x\right]<\infty$ and so, we conclude that l is an attainable state. In case (b), the second term of the r.h.s. of (3.34) is infinite and so, $\mathbb{E}\left[T_{l,b}\middle|Z=x\right]=\infty$ and we conclude that l is an unattainable state. For case (c), we hold that

$$\mathbb{E}\left[T_{l,b}\middle|Z=x\right]=2\int_x^b \frac{S(b)-S(z)}{S'(z)\,F^2(z)}dz$$

As l is a non-attracting state, $S(b)-S(z)=\infty$ for all $z\in(l,b)$. This implies that $\mathbb{E}\left[T_{l,b}\middle|Z=x\right]=\infty$ because $S'(z)\,F^2(z)>0$ for all $z\in\mathbb{R}$. □

In the same way, we present the definition of attainable right state r and its characterization.

Definition 3.25. Let $r\in\mathbb{R}$ and $a\in(-\infty,r)$. We said

- r is an *attainable state* if for all $x\in(a,r)$, $\mathbb{E}\left[T_{a,r}\middle|Z=x\right]<\infty$.
- r is an *unattainable state* if for all $x\in(a,r)$, $\mathbb{E}\left[T_{a,r}\middle|Z=x\right]=\infty$.

Proposition 3.26. *(a) If r is an attracting state, then r is an attainable state if*

$$\lim_{a\to r^-}\int_x^a \frac{S(a)-S(z)}{S'(z)\,F^2(z)}dz<\infty$$

for some $x<r$.

(b) If r is an attracting, then r is an unattainable state if

$$\lim_{a\to r^-}\int_x^a \frac{S(a)-S(z)}{S'(z)\,F^2(z)}dz=\infty$$

for some $x<r$.

(c) If r is a non-attracting state, then r is an unattainable state.

Let finalize this chapter with the next proposition.

Proposition 3.27. *Let $H(x)$ be twice continuously differentiable and monotone in x and let $X(t),t\in[t_0,\infty)$ be the solution of the SDE (3.30) that is a diffusion. We hold the next cases:*

- *If l is an attracting state of the process $X(t),t\in[t_0,\infty)$, then $H(l)$ is an attracting state of the process $H(X(t)),t\in[t_0,\infty)$.*
- *If l is a non-attracting state of the process $X(t),t\in[t_0,\infty)$, then $H(l)$ is a non-attracting state of the process $H(X(t)),t\in[t_0,\infty)$.*
- *If l is an attainable state of the process $X(t),t\in[t_0,\infty)$, then $H(l)$ is an attainable state of the process $H(X(t)),t\in[t_0,\infty)$.*

- *If l is an unattainable state of the process $X(t), t \in [t_0, \infty)$, then $H(l)$ is an unattainable state of the process $H(X(t)), t \in [t_0, \infty)$.*

Proof. As $H(x)$ is twice continuously differentiable and monotone function, then H^{-1} exists. By this,

$$\left[H\left(X\left(T_{H(l),H(b)}\right)\right) = H(l)\,|Z = x\right] = \left[X\left(T_{l,b}\right) = l|Z = x\right]$$

and so,

$$P\left[H\left(X\left(T_{H(l),H(b)}\right)\right) = H(l)\,|Z = x\right] = P\left[X\left(T_{l,b}\right) = l|Z = x\right].$$

Let consider that l is an attracting state. Then,

$$P\left[H\left(X\left(T_{l,b}\right)\right) = H(l)\,|Z = x\right] = P\left[X\left(T_{l,b}\right) = l|Z = x\right] > 0$$

and we conclude that $H(l)$ is an attracting state. On the other hand, if l is a non-attracting state, then

$$P\left[H\left(X\left(T_{H(l),H(b)}\right)\right) = H(l)\,|Z = x\right] = P\left[X\left(T_{l,b}\right) = l|Z = x\right] = 0$$

which implies that $H(l)$ is a non-attracting state.

By other hand, let observe that $T_{H(l),H(b)} = T_{l,b}$ because $H(x)$ is monotone. Thus,

$$\mathbb{E}\left[T_{H(l),H(b)}|Z = x\right] = \mathbb{E}\left[T_{l,b}|Z = x\right].$$

Now, if l is an attainable state, then

$$\mathbb{E}\left[T_{H(l),H(b)}\middle|Z = x\right] = \mathbb{E}\left[T_{l,b}\middle|Z = x\right] < \infty,$$

implying that $H(l)$ is an attainable state of the process $H(X(t)), t \geq t_0$. In the same way, if l is an unattainable state, then $H(l)$ is an unattainable state of the process $H(X(t)), t \geq t_0$ because

$$\mathbb{E}\left[T_{H(l),H(b)}\middle|Z = x\right] = \mathbb{E}\left[T_{l,b}\middle|Z = x\right] = \infty.$$

□

Chapter 4

The controversy: Itô or Stratonovich?

Introduction

This chapter discusses the controversy Itô vs Stratonovich for a class of population growth models following Braumann's papers [3, 4]. In [4], Braumann resolves the controversy in two cases: in the first one it is assumed the population model has a *density-independent* average growth rate, while in the section one it is assumed the average growth rate is *density-dependent*. In [3], Braumann extends the analysis to population models with harvesting.

4.1 The controversy

Consider a population that evolves without interacting with other species but being subject to environmental noise or random perturbations. To model the population dynamic, one can begin assuming that the population evolves according to the ordinary differential equation

$$dN(t) = r(t)\,N(t)\,dt$$

with initial condition $N(0) = N_0$, where $N(t)$ is the population size and $r(t)$ is the (per-capita or relative) growth rate of the population $N(t)$ at time $t \geq 0$. Next, to take into account the random perturbations, one can assume that the growth rate has the form

$$r(t) = g(t) + \sigma(t)\,W(t) \tag{4.1}$$

where $g : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are "deterministic" functions, and the process $W(t), t \geq 0$ is formed by independent identically distributed random variables with zero mean and finite variance. Thus, the population model becomes in

$$dN(t) = g(t)\,N(t)\,dt + \sigma(t)\,N(t)\,W(t)\,dt, \tag{4.2}$$

which, proceeding formally, can be rewritten as

$$N(t) = N(0) + \int_0^t g(s)\,N(s)ds + \int_0^t \sigma(s)\,N(s)\,W(s)ds. \tag{4.3}$$

Concerning to the last equations two comments are in order. On one hand, it is well known the process $W(t), t \geq 0$ does not exists as a proper stochastic process and it is formally regarded as the "derivative" of the Brownian motion $B(t), t \geq 0$ (for more details about proper stochastic process, see [7]); this latter fact is expressed by writing

$$dB(t) = W(t)\,dt.$$

Thus, equations (4.2) and (4.3) now read as

$$dN(t) = g(t)\,N(t)\,dt + \sigma(t)\,N(t)\,dB(t)\,, \tag{4.4}$$

and

$$N(t) = N(0) + \int_0^t g(s)\,N(s)ds + \int_0^t \sigma(s)\,N(s)\,dB(s)\,. \tag{4.5}$$

On the other hand, the term

$$\int_0^t \sigma(s)\,N(s)\,dB(s) \tag{4.6}$$

is usually interpreted either as an Itô integral or as a Stratonovich integral; thus, the properties of the process $N(t), t \geq 0$, will obviously depend on the integral used. This leads us to the core of the controversy: for instance, one can wonder whether the Itô calculus describes better the long-term population behaviour than the Stratonovich calculus, or vice versa.

We show below how these calculi predicts different faith for the population analysing a particular case of model (4.4) or (4.5). To avoid confusions, the customary notation (4.6) is reserved for the Itô integral, while we will write

$$\int_0^t \sigma(s)\,N(s) \circ dB(s) \tag{4.7}$$

for the Stratonovich integral.

Define the process

$$Y(t) := \ln N(t)\,, \quad \text{and} \quad Y_0 := \ln N_0. \tag{4.8}$$

The case of a deterministic model. Suppose that $g(t) = g$ for all $t \geq 0$, where g is a constant, and also that $\sigma \equiv 0$; thus, the population evolves according to the ordinary differential equation

$$dN(t) = gN(t)\,dt,\ t \geq 0, \quad \text{and} \ \ N(0) = N_0 > 0,$$

which can be equivalently rewritten as

$$dY(t) = gdt,\ t \geq 0, \quad \text{and} \ \ Y(0) = Y_0 > 0.$$

Then,

$$Y(t) = Y_0 + gt,$$

which implies that

$$N(t) = N_0 \exp gt. \tag{4.9}$$

Remark 9. The long-term population behaviour is determined by the parameter g as follows:

(i) if $g > 0$, then $\lim_{t\to\infty} N(t) = \infty$;

(ii) if $g = 0$, then $N(t) = N_0$ for all $t \geq 0$;

(iii) if $g < 0$, then $\lim_{t\to\infty} N(t) = 0$.

To verify these fact hold true, observe that

$$\lim_{t\to\infty} \exp\{gt\} = \begin{cases} \infty & \text{if} \quad g > 0, \\ 1 & \text{if} \quad g = 0, \\ 0 & \text{if} \quad g > 0 \end{cases}$$

Thus,

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} N_0 \exp\{gt\} = \begin{cases} \infty & \text{if} \quad g > 0, \\ N_0 & \text{if} \quad g = 0, \\ 0 & \text{if} \quad g > 0 \end{cases}$$

The case of the Itô calculus. Suppose that $g(t) = g$ and $\sigma(t) = \sigma$ for all $t \geq 0$, where $g \in \mathbb{R}$ and $\sigma > 0$. Thus, the process $N(t), t \geq 0$ satisfies the stochastic differential equation

$$dN(t) = gN(t)\,dt + \sigma N(t)\,dB(t), \tag{4.10}$$

with $N(0) = N_0$. Then, by the Itô formula (2.21), the process (4.10) satisfies the equation

$$dY(t) = \left(g - \frac{\sigma^2}{2}\right) dt + \sigma dB(t).$$

Hence, the Itô calculus yields

$$Y(t) = Y(0) + \left(g - \frac{\sigma^2}{2}\right) t + \sigma B(t),$$

which in turn implies that

$$N(t) = N_0 \exp\left\{ \left(g - \frac{\sigma^2}{2}\right) t + \sigma B(t) \right\} \forall t \geq 0. \tag{4.11}$$

Now observe that $Y(t)$ is normally distributed with

$$\mathbb{E}[Y(t)] = Y_0 + \left(g - \frac{\sigma^2}{2}\right) t \quad \text{and} \quad Var[Y(t)] = \sigma^2 t;$$

hence, $N(t)$ has log-normal distribution with

$$\mathbb{E}[N(t)] = N_0 \exp\left\{gt\right\} \quad \text{and} \quad Var[N(t)] = N_0^2 \left[\exp\left\{\sigma^2 t\right\} - 1\right] \exp\left\{2gt\right\}.$$

The next remark describes the long-term behaviour of the process (4.10).

Remark 10. (a) if $g > \frac{\sigma^2}{2}$, then $\lim_{t\to\infty} N(t) = \infty$ a.s.;

(b) if $g = \frac{\sigma^2}{2}$, then $\liminf_{t\to\infty} N(t) = 0$ and $\limsup_{t\to\infty} N(t) = \infty$ a.s.;

(c) if $g < \frac{\sigma^2}{2}$, then $\lim_{t\to\infty} N(t) = 0$ a.s.;

The case of Stratonovich calculus. Suppose again that $g(t) = a$ and $\sigma(t) = \sigma$ for all $t \geq 0$, but now consider the Stratonovich integral. Then, the process $N(t), t \geq 0$ satisfies the stochastic differential equation

$$dN(t) = gN(t)\,dt + \sigma N(t) \circ dB(t), \tag{4.12}$$

with $N(0) = N_0$, which can be equivalently rewritten by the definition of the Stratonovich integral (2.22) as

$$dN(t) = \left(g + \frac{\sigma^2}{2}\right) N(t)\,dt + \sigma N(t)\,dB(t), \tag{4.13}$$

with the same initial condition. Then, by Itô formula (2.21), the process (4.12) satisfies the equation

$$dY(t) = gdt + \sigma dB(t).$$

Hence, the Stratonovich calculus yields

$$Y(t) = Y(0) + gt + \sigma B(t),$$

which in turn implies that

$$N(t) = N_0 \exp\left\{gt + \sigma B(t)\right\} \forall t \geq 0. \tag{4.14}$$

Now observe that $Y(t)$ is normally distributed with

$$\mathbb{E}[Y(t)] = Y_0 + gt \quad \text{and} \quad Var[Y(t)] = \sigma^2 t;$$

hence, $N(t)$ has log-normal distribution with

$$\begin{aligned} \mathbb{E}[N(t)] &= N_0 \exp\left\{\left(g + \frac{\sigma^2}{2}\right)t\right\} \quad \text{and} \\ Var[N(t)] &= N_0^2 \left[\exp\left\{\sigma^2 t\right\} - 1\right] \exp\left\{\left(2g + \sigma^2\right)t\right\}. \end{aligned}$$

The next remark describes the long-term behaviour of the process (4.12).

Remark 11. (a) if $g > 0$, then $\lim_{t\to\infty} N(t) = \infty$ a.s.;

(b) if $g = 0$, then $\liminf_{t\to\infty} N(t) = 0$ and $\limsup_{t\to\infty} N(t) = \infty$ a.s.;

(c) if $g < 0$, then $\lim_{t\to\infty} N(t) = 0$ a.s.;

4.2 The resolution of the controversy: the density-independent growth rate case

As was discussed above, in both calculi the long-term behaviour of population depends on the growth rate parameter g. We will take a closer look on such parameter. First, we consider the deterministic model

$$dN(t) = r(N(t))\, N(t)\, dt.$$

In many cases, the growth rate $r(t)$ can be suitable modelled as a function of the current size population, that is,

$$r(t) = R(N(t)),\ t \geq 0,$$

where the growth rate function R is a function from $\mathbb{R}_+$ into $\mathbb{R}$. Then,

$$\begin{aligned} R(x) &= \frac{1}{x}\frac{dN(t)}{dt}\Big|_{N(t)=x} \\ &= \frac{1}{x}\lim_{h\to 0}\frac{N(t+h)-x}{h}. \end{aligned}$$

Definition 4.1. Consider a stochastic population process $N(t), t \geq 0$.

(a) The *arithmetic average growth rate function* of $N(t), t \geq 0$, is defined as

$$R_a(x) = \frac{1}{x}\lim_{h\to 0}\frac{\mathbb{E}[N(t+h)\,|N(t)=x]-x}{h},\ x>0; \tag{4.15}$$

(b) The *geometric average growth rate function* is defined as

$$R_g(x) = \frac{1}{x}\lim_{h\to 0}\frac{\exp(\mathbb{E}\{\ln[N(t+h)]\,|N(t)=x\})-x}{h},\ x>0. \tag{4.16}$$

Proposition 4.2. *Suppose that the population evolves according to the Itô stochastic differential equation*

$$dN(t) = gN(t)\,dt + \sigma N(t)\,dB(t),\ \ N(0) = N_0.$$

Then:

(a) $R_a(x) = g$ *for all* $x > 0$*;*

(b) $R_g(x) = g - \frac{\sigma^2}{2}$ *for all* $x > 0$.

Proof. Firstly note that

$$\begin{aligned} N(t+h) &= N_0 \exp\left\{\left(g-\frac{\sigma^2}{2}\right)(t+h)+\sigma B(t+h)\right\} \\ &= N_0 \exp\left\{\left(g-\frac{\sigma^2}{2}\right)t+\sigma B(t)\right\}\times \\ &\quad \exp\left\{\left(g-\frac{\sigma^2}{2}\right)h+\sigma[B(t+h)-B(t)]\right\} \\ &= N(t)\exp\left\{\left(g-\frac{\sigma^2}{2}\right)h+\sigma[B(t+h)-B(t)]\right\} \end{aligned}$$

Next, observe that

$$\begin{aligned}
R_a(x) &= \frac{1}{x}\lim_{h\to 0^+}\frac{\mathbb{E}\left[N(t+h)\middle|N(t)=x\right]-x}{h}\\
&= \frac{1}{x}\lim_{h\to 0^+}\frac{\mathbb{E}\left[N(t)\exp\left\{\left(g-\frac{\sigma^2}{2}\right)h+\sigma\left[B(t+h)-B(t)\right]\right\}\middle|N(t)=x\right]-x}{h}\\
&= \frac{1}{x}\lim_{h\to 0^+}\frac{x\exp\left\{\left(g-\frac{\sigma^2}{2}\right)h\right\}\mathbb{E}\left[\exp\left\{\sigma\left[B(t+h)-B(t)\right]\right\}\middle|N(t)=x\right]-x}{h}\\
&= \lim_{h\to 0^+}\frac{\exp\left\{\left(g-\frac{\sigma^2}{2}\right)h\right\}\mathbb{E}\left[\exp\left\{\sigma\left[B(t+h)-B(t)\right]\right\}\right]-1}{h}\\
&= \lim_{h\to 0^+}\frac{\exp\left\{\left(g-\frac{\sigma^2}{2}\right)h\right\}\exp\left\{\frac{\sigma^2}{2}h\right\}-1}{h}\\
&= \lim_{h\to 0^+}\frac{\exp[gh]-1}{h}\\
&= g
\end{aligned}$$

Now to compute the geometric growth, notice that

$$\begin{aligned}
\ln[N(t+h)] &= Y(t+h)\\
&= Y(0)+\left(g-\frac{\sigma^2}{2}\right)(t+h)+\sigma B(t+h)\\
&= Y(0)+\left(g-\frac{\sigma^2}{2}\right)t+\sigma B(t)+\left(g-\frac{\sigma^2}{2}\right)h+\sigma\left[B(t+h)-B(t)\right]\\
&= Y(t)+\left(g-\frac{\sigma^2}{2}\right)h+\sigma\left[B(t+h)-B(t)\right]\\
&= \ln[N(t)]+\left(g-\frac{\sigma^2}{2}\right)h+\sigma\left[B(t+h)-B(t)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
R_g(x) &= \frac{1}{x}\lim_{h\to 0^+}\frac{\exp\left(\mathbb{E}\left\{\ln[N(t+h)]\,|N(t)=x\right\}\right)-x}{h}\\
&= \frac{1}{x}\lim_{h\to 0^+}\frac{\exp\left(\mathbb{E}\left\{\ln[N(t)]+\left(g-\frac{\sigma^2}{2}\right)h+\sigma\left[B(t+h)-B(t)\right]|N(t)=x\right\}\right)-x}{h}\\
&= \frac{1}{x}\lim_{h\to 0^+}\frac{\exp\left(\ln(x)+\left(g-\frac{\sigma^2}{2}\right)h+\sigma\mathbb{E}\left[B(t+h)-B(t)\middle|N(t)=x\right]\right)-x}{h}\\
&= \frac{1}{x}\lim_{h\to 0^+}\frac{x\exp\left(\left(g-\frac{\sigma^2}{2}\right)h+\sigma\mathbb{E}\left[B(t+h)-B(t)\right]\right)-x}{h}\\
&= \lim_{h\to 0}\frac{\exp\left(\left(g-\frac{\sigma^2}{2}\right)h\right)-1}{h}\\
&= g-\frac{\sigma^2}{2}.
\end{aligned}$$

□

Proposition 4.3. *Suppose that the population evolves according to the Stratonovich stochastic differential equation*

$$dN(t)=gN(t)\,dt+\sigma N(t)\circ dB(t).$$

Then:

(a) $R_a(x) = g + \frac{\sigma^2}{2}$ *for all* $x > 0$*;*

(b) $R_g(x) = g$ *for all* $x > 0$*.*

Proof. By the definition of the Stratonovich integral (2.22), the last SDE is equivalent to the Itô SDE

$$dN(t) = \left(g + \frac{\sigma^2}{2}\right) N(t)\, dt + \sigma N(t)\, dB(t).$$

and using Proposition 4.2, we have that

$$\begin{aligned} R_a(x) &= g + \frac{\sigma^2}{2} \\ R_g(x) &= g + \frac{\sigma^2}{2} - \frac{\sigma^2}{2} = g. \end{aligned}$$

□

Remark 12. Notice that in both cases, the relation $R_a(x) = R_g(x) + \frac{\sigma^2}{2}$ holds true.

The computation of both averages present a solution to the controversy. Replacing the unspecified average growth rate g by the specified average growth rate of each case, we obtain that the solution of the Itô SDE

$$dN(t) = gN(t)\, dt + \sigma N(t)\, dB(t).$$

is

$$\begin{aligned} N(t) &= N_0 \exp\left\{\left[R_a - \frac{\sigma^2}{2}\right] t + \sigma B(t)\right\} \\ &= N_0 \exp\{R_g t + \sigma B(t)\} \end{aligned}$$

whereas the solution of the Stratonovich SDE

$$dN(t) = gN(t)\, dt + \sigma N(t) \circ dB(t).$$

is

$$N(t) = N_0 \exp\{R_g t + \sigma B(t)\}$$

which implies that both solutions are the same one. Therefore, it is indifferent which calculus is used to describe the long-term behaviour of the population, whenever one considers the right average growth rate: arithmetic growth for the Itô calculus and geometric growth for the Stratonovich calculus.

The long-term behaviour of the process $N(t)$ can be expressed as next:

Remark 13. (a) If $R_g(x) > 0$, then $\lim_{t\to\infty} N(t) = \infty$ a.s.;

(b) if $R_g(x) = 0$, then $\liminf_{t\to\infty} N(t) = 0$ and $\limsup_{t\to\infty} N(t) = \infty$ a.s.;

(c) if $R_g(x) < 0$, then $\lim_{t\to\infty} N(t) = 0$ a.s.;

4.3 The resolution of the controversy: the density-dependent growth rate case

In the discussion of the last section, the controversy between both calculi was resolved for the density-independent growth rate case. Now, its turn to analyse the controversy in the density-dependent growth rate case, where $g_i(x)$ will denote average growth rate for the Itô case and $g_s(x)$ will denote the average growth rate for the Stratonovich case.

The Itô case

Consider the Itô stochastic differential equation

$$dN(t) = g_i(N(t))\, N(t)\, dt + \sigma N(t)\, dB(t)\,, \tag{4.17}$$

with initial condition $N(0) = N_0 > 0$ and let $G_i(x) := x g_i(x)$ and $\Sigma(x) := \sigma x$. Assume that the following conditions hold:

(A) $g_i : (0, \infty) \to \mathbb{R}$ is a continuously differentiable function and;

(B) the limits $g_i(0^+) := \lim_{N \to 0^+} g_i(N)$, $G_i(0^+) := \lim_{N \to 0^+} G_i(N)$ exists and, independent of the value of $g_i(0^+)$, $G_i(0^+) = 0$.

Then, $G_i(x)$ and $\Sigma(x)$ are continuously differentiable functions and, by Theorem 3.9, there exists a global solution $N(t), t \geq 0$ for the SDE (4.17) until the time-explosion τ.

If we want to avoid this explosion, $N = -\infty$ and $N = \infty$ cannot be reached in finite time. So, let assume further that

(C) the boundaries $N = 0$ and $N = \infty$ are unattainable states of the process $N(t), t \geq 0$.

This assumption implies that process can not reaches any state $N < 0$ if $N_0 > 0$ and thus, $N = -\infty$ is also an unattainable state. Therefore, there is no explosion and the solution $N(t), t \geq 0$ of the SDE (4.17) exists for all $t \geq 0$ and has values on $(0, \infty)$.

On the other hand, $N(t), t \geq 0$ is not necessary a diffusion process. So, we need to hold two more conditions relative to the moments of the process $N(t), t \geq 0$ and the rescaled process $Y(t) = \ln[N(t)], t \geq 0$. These assumptions are:

(D) the limits

$$\lim_{h \to 0^+} \frac{\mathbb{E}\left[N(t+h) - x \middle| N(t) = x\right]}{h}; \quad \lim_{h \to 0^+} \frac{\mathbb{E}\left[|N(t+h) - x|^2 \middle| N(t) = x\right]}{h} \tag{4.18}$$

exists,

(E) the limits

$$\lim_{h\to 0^+}\frac{\mathbb{E}\left[\ln\left(N\left(t+h\right)\right)-\ln\left(x\right)\Big|N\left(t\right)=x\right]}{h};\ \lim_{h\to 0^+}\frac{\mathbb{E}\left[\left|\ln\left(N\left(t+h\right)\right)-\ln\left(x\right)\right|^2\Big|N\left(t\right)=x\right]}{h} \tag{4.19}$$

exists.

Assumption (D) implies the existence of the arithmetic average growth rate (4.15). In other way, the Itô formula applied to $Y(t)=\ln(N(t))$ implies that the SDE (4.17) is changed to

$$dY\left(t\right)=\left(g_i\left(e^{Y(t)}\right)-\frac{\sigma^2}{2}\right)dt+\sigma dB\left(t\right),\ \ Y\left(0\right)=Y_0=\ln\left(N_0\right). \tag{4.20}$$

Next, if M is the first limit on assumption (E), let observe that

$$\begin{aligned}\exp\{\mathbb{E}\left[\ln\left(N\left(t+h\right)\right)\Big|N\left(t\right)=x\right]\} &= \exp\{\ln\left(x\right)+Mh+o\left(h\right)\}\\ &= x\exp\{Mh+o\left(h\right)\}\\ &= x\left(1+Mh+o\left(h\right)\right).\end{aligned}$$

Thus,

$$\begin{aligned}R_g\left(x\right) &= \frac{1}{x}\lim_{h\to 0}\frac{\exp\{\mathbb{E}\left[\ln\left(N\left(t+h\right)\right)\Big|N\left(t\right)=x\right]\}-x}{h}\\ &= \frac{1}{x}\lim_{h\to 0}\frac{x\left(1+Mh+o\left(h\right)\right)-x}{h}\\ &= \lim_{h\to 0}\frac{Mh+o\left(h\right)}{h}=M.\end{aligned}$$

By this, assumption (E) implies the existence of the geometric growth rate. These conditions are resumed in the next theorem, proposed by Braumann in [4].

Theorem 4.4. *Let $g_i(x)$ be a function that satisfies assumptions (A) and (B) and let $N(t)$ be the solution of the SDE*

$$dN\left(t\right)=N\left(t\right)g_i\left(N\left(t\right)\right)dt+\sigma N\left(t\right)dB\left(t\right)$$

If assumption (C) is satisfied, then $N(t)$ exists for all $t\geq 0$. Moreover,

- *if assumption (D) is satisfied, the arithmetic growth rate exists and*

$$R_a\left(x\right)=g_i\left(x\right).$$

- *if assumption (E) is satisfied, the geometric growth rate exists and*

$$R_g\left(x\right)=g_i\left(x\right)-\frac{\sigma^2}{2}.$$

Assumptions (A) and (B) sounds so realistic (in particular, (B) not permit spontaneous generation). But, what is about the restrictive assumptions (C), (D) and (E)? Let show more conditions to warranty these assumptions.

(F) There are $A \in (-\infty, \infty)$ and $B \in (0, \infty)$ such that $g_i(x) \leq A$ for $x > B$.

(G) If it happens that $g_i(0^+) = \infty$, then we must have $g_i(x) \leq C\left(1 + \frac{1}{x^2}\right)$ (with C some positive finite constant) for $x \in (0, \delta)$ (with $\delta > 0$).

(H) If it happens that $g_i(0^+) = -\infty$, then we must have $g_i(x) \geq D \ln(x)$ (with D some positive finite constant) for $x \in (0, \delta)$ (with $\delta > 0$).

With these assumptions, including (A) and (B), we hold the next three lemmas, that provides the existence and uniqueness of solutions for the SDE (4.17) and (4.20).

Lemma 4.5. *Let $g_i(x)$ be a function that satisfies assumptions (A) and (B).*

(i) If assumptions (F) is satisfied, then there exists a constant $K_1 > 0$ such that, for all $x > 0$

$$x^2 g_i(x) \leq K_1\left(1 + x^2\right). \tag{4.21}$$

(ii) If assumptions (F) and (G) is satisfied, then there exists a constant $K_2 > 0$ such that, for all $x > 0$

$$x g_i(x) \leq K_2\left(1 + x^2\right). \tag{4.22}$$

(iii) If assumptions (F) and (H) are satisfied, then there exists a constant $K_3 > 0$ such that, for all $x > 0$,

$$\ln(x)\left[g_i(x) - \frac{\sigma^2}{2}\right] \leq K_3\left(1 + \ln(x)^2\right). \tag{4.23}$$

(iv) If assumptions (F) and (G) are satisfied, then there exists a constant $K_4 > 0$ such that, for all $x > 0$,

$$g_i(x) - \frac{\sigma^2}{2} \leq K_4\left(1 + \ln(x)^2\right). \tag{4.24}$$

Proof. Proof of (i). By assumption (F), $g_i(x) \leq A$ for $x \in (B, \infty)$. Then,

$$x^2 g_i(x) \leq A x^2 < A\left(1 + x^2\right) \text{ for all } x \in (B, \infty).$$

Next, assumption (B) implies that for a fix $\epsilon > 0$, there exists $\delta > 0$ such that $|x g_i(x)| \leq \epsilon$ for $x < \delta$. This implies that

$$x^2 g_i(x) \leq x|x g_i(x)| \leq \epsilon x < \epsilon\left(1 + x^2\right) \text{ for all } x \in (0, \delta).$$

Later, assumption (A) and Weierstrass' theorem implies that $g_i(x)$ attaches its maximum M on $[\delta, B]$. Thus,

$$x^2 g_i(x) \leq M x^2 < M\left(1 + x^2\right) \text{ for all } x \in [\delta, B].$$

Finally, if $K_1 = \max\{\epsilon, M, A\}$, then the inequality (4.21) is satisfied for all $x > 0$.

Proof of (ii). By assumption (F), $g_i(x) \leq A$ for $x \in (B, \infty)$. Then,

$$xg_i(x) \leq Ax < A\left(1+x^2\right) \text{ for all } x \in (B, \infty).$$

Next, assumption (B) implies that for a fix $\epsilon > 0$, there exists $\delta > 0$ such that $|xg_i(x)| \leq \epsilon$ for $x < \delta$. This implies that

$$xg_i(x) \leq |xg_i(x)| \leq \epsilon < \epsilon\left(1+x^2\right) \text{ for all } x \in (0, \delta).$$

Later, assumption (A) and Weierstrass' theorem implies that $g_i(x)$ attaches its maximum M on $[\delta, B]$. Thus,

$$xg_i(x) \leq Mx < 2M\left(1+x^2\right) \text{ for all } x \in [\delta, B].$$

Finally, if $K_2 = \max\{\epsilon, 2M, A\}$, then the inequality (4.22) is satisfied for all $x > 0$.

Proof of (iii). Let $\tilde{B} = \max\{B, \exp\}$. By assumption (F), $\ln(x) > 1$ and $g_i(x) - \frac{\sigma^2}{2} \leq A - \frac{\sigma^2}{2}$, with $A - \frac{\sigma^2}{2} > 0$ for all $x \geq \tilde{B}$. Thus,

$$\begin{aligned} \ln(x)\left(g_i(x) - \frac{\sigma^2}{2}\right) &\leq \left(A - \frac{\sigma^2}{2}\right)\ln(x) \\ &\leq \left(A - \frac{\sigma^2}{2}\right)\left(1 + \ln(x)^2\right) \end{aligned}$$

for all $x > \tilde{B}$. By other hand, assumption (B) implies the existence of the limit $W = \lim_{x\to 0^+} g_i(x) - \frac{\sigma^2}{2}$.

- If W is finite, given $\eta > 0$ there exists $\delta_1 < \exp^{-1}$ such that $|g_i(x) - \frac{\sigma^2}{2} - W| < \eta$ if $x < \delta_1$. Thus,

$$\begin{aligned} \left|\ln(x)\left(g_i(x) - \frac{\sigma^2}{2}\right)\right| &\leq \left|\ln(x)\left(g_i(x) - \frac{\sigma^2}{2}\right) - W\ln(x)\right| \\ &\quad + |W\ln(x)| \\ &\leq |\eta\ln(x)| + |W\ln(x)| \\ &= (|W| + \eta)|\ln(x)| \\ &\leq (|W| + \eta)\left(1 + \ln(x)^2\right). \end{aligned}$$

- If $W = \infty$, there exists $\delta_2 < \exp^{-1}$ such that $g_i(x) - \frac{\sigma^2}{2} > 0$ for $x < \delta_2$. Thus, $\ln(x)\left(g_i(x) - \frac{\sigma^2}{2}\right) < 0 < 1 + \ln(x)^2$.

- If $W = -\infty$, there exists $\delta_3 < \exp^{-1}$ such that $0 > g_i(x) - \frac{\sigma^2}{2}$.

By assumption (H)

$$\begin{aligned}
\ln(x)\left(g_i(x)-\frac{\sigma^2}{2}\right) &\leq \ln(x)\left(D\ln(x)-\frac{\sigma^2}{2}\right)\\
&= D\ln(x)^2+\frac{\sigma^2}{2}|\ln(x)|\\
&\leq \left(D+\frac{\sigma^2}{2}\right)\ln(x)^2\\
&< \left(D+\frac{\sigma^2}{2}\right)\left(1+\ln(x)^2\right).
\end{aligned}$$

Finally, define $\tilde{\delta}=\min\{\delta_1,\delta_2,\delta_3\}$. Thus, assumption (A) and Weierstrass' theorem implies that $g_i(x)-\frac{\sigma^2}{2}$ attaches its maximum $\tilde{M}$ on $\left[\tilde{\delta},\tilde{B}\right]$. Thus,

$$\begin{aligned}
\ln(x)\left(g_i(x)-\frac{\sigma^2}{2}\right) &\leq \tilde{M}|\ln(x)|\\
&\leq \tilde{M}\left(1+|\ln(x)|\right)^2\\
&\leq 2\tilde{M}\left(1+\ln(x)^2\right).
\end{aligned}$$

Therefore, if $K_3=\max\{A-\frac{\sigma^2}{2},|W|+\eta,1,D+\frac{\sigma^2}{2},2\tilde{M}\}$, then the inequality (4.23) is satisfied for all $x>0$.

Proof of (iv). Let $\tilde{B}=\max\{B,\exp\}$. By assumption (F), $g_i(x)-\frac{\sigma^2}{2}\leq A-\frac{\sigma^2}{2}$, with $A-\frac{\sigma^2}{2}>0$ for all $x\geq\tilde{B}$. Thus,

$$g_i(x)-\frac{\sigma^2}{2}\leq A-\frac{\sigma^2}{2}\leq\left(A-\frac{\sigma^2}{2}\right)\left(1+\ln(x)^2\right)\ \text{ for all } x>\tilde{B}.$$

By other hand, assumption (B) implies the existence of the limit $W=\lim_{x\to 0^+}g_i(x)-\frac{\sigma^2}{2}$.

- If W is finite, given $\eta>0$ there exists $\delta_1<\exp^{-1}$ such that $|g_i(x)-\frac{\sigma^2}{2}-W|<\eta$ if $x<\delta_1$. Thus,

$$\begin{aligned}
\left|\left(g_i(x)-\frac{\sigma^2}{2}\right)\right| &\leq \left|\left(g_i(x)-\frac{\sigma^2}{2}\right)-W\right|+|W|\\
&\leq (|W|+\eta)\\
&\leq (|W|+\eta)\left(1+\ln(x)^2\right)\ \text{ for all } x<\delta_1.
\end{aligned}$$

- If $W=\infty$, there exists $\delta_2<\exp^{-1}$ such that $g_i(x)-\frac{\sigma^2}{2}>0$ for $x<\delta_2$. Thus, by assumption (G),

$$\begin{aligned}
g_i(x)-\frac{\sigma^2}{2} &< g_i(x)\\
&\leq C\left(1+\frac{1}{x^2}\right)\\
&\leq C\left(1+\ln(x)^2\right)\ \text{ for all } x<\delta_2.
\end{aligned}$$

- If $W=-\infty$, there exists $\delta_3<\exp^{-1}$ such that $0>g_i\left(x\right)-\frac{\sigma^2}{2}$. Thus,

$$g_i\left(x\right)-\frac{\sigma^2}{2}<0<1+\ln\left(x\right)^2 \text{ for all } x<\delta_3.$$

Finally, define $\tilde{\delta}=\min\{\delta_1,\delta_2,\delta_3\}$. Thus, assumption (A) and Weierstrass' theorem implies that $g_i\left(x\right)-\frac{\sigma^2}{2}$ attaches its maximum $\tilde{M}$ on $\left[\tilde{\delta},\tilde{B}\right]$. Thus,

$$g_i\left(x\right)-\frac{\sigma^2}{2}\le\tilde{M}\le\tilde{M}\left(1+\ln\left(x\right)^2\right) \text{ for all } x\in\left[\tilde{\delta},\tilde{B}\right].$$

Therefore, if $K_4=\max\{A-\frac{\sigma^2}{2},C,|W|+\eta,1,\tilde{M}\}$, then the inequality (4.24) is satisfied for all $x>0$.

□

Lemma 4.6. *Let $g_i\left(x\right)$ be a function that satisfies assumptions (A), (B) and (H). If $W=\lim_{x\to0^+}g_i\left(x\right)-\frac{\sigma^2}{2}<0$, then there exists $\beta,D>0$ such that*

$$\left(D+\frac{\sigma^2}{2}\right)\ln\left(x\right)\le g_i\left(x\right)-\frac{\sigma^2}{2}\le-\beta \tag{4.25}$$

for $x\in(0,\delta)$ and $\delta>0$.

Proof. Suppose that $W=-\infty$ and let choose $\delta<e^{-1}$. By assumption (H), $g_i\left(x\right)\ge D\ln\left(x\right)$ for $x\in(0,\delta)$ with $D>0$. Thus,

$$g_i\left(x\right)-\frac{\sigma^2}{2}\ge D\ln\left(x\right)-\frac{\sigma^2}{2}\ge\left(D+\frac{\sigma^2}{2}\right)\ln\left(x\right)$$

for $x\in(0,\delta)$. By other hand, for each $\beta>0$ there exists $\delta_1>0$ such that $g_i\left(x\right)-\frac{\sigma^2}{2}\le-\beta$ if $x<\delta_1$. If we choose $\delta_2=\min\{\delta,\delta_1\}$, then for each $x\in(0,\delta_2)$, (4.25) is satisfied.

In the case that W is finite, there exists $\alpha,\beta>0$ such that $-\alpha\le g_i\left(x\right)-\frac{\sigma^2}{2}\le-\beta$ for each $x\in(0,\delta)$. Moreover, if we choose $x\le\exp\left\{-\frac{\alpha}{D+\frac{\sigma^2}{2}}\right\}$, then

$$\left(D+\frac{\sigma^2}{2}\right)\ln\left(x\right)\le-\alpha.$$

Thus, (4.25) is satisfied, as we want to show. □

Theorem 4.7. *Let suppose that $g_i(x)$ satisfies the assumptions (A), (B), (F), (G) and (H). Then, there exists a unique solution $N(t), t \geq 0$ for the SDE*

$$dN(t) = g_i(N(t))\, N(t)\, dt + \sigma N(t)\, dB(t)$$

with initial condition $N(0) = N_0 > 0$. Further that, assumptions (C), (D) and (E) are satisfied and so, by Theorem 4.4:

- *$N(t)$ is a diffusion process on $(0, \infty)$,*
- *the arithmetic average growth rate $R_a(x)$ exists and $R_a(x) = g_i(x)$,*
- *the geometric average growth rate $R_g(x)$ exists and $R_g(x) = g_i(x) - \frac{\sigma^2}{2}$.*

Proof. By the part (i) of Lemma 4.5, the coefficients $G_i(x) = xg_i(x)$ and $\Sigma(x) = \sigma x$ satisfies the conditions of existence and uniqueness of Theorem 3.4 for the SDE (4.17) up to an explosion time $\tau \leq \infty$. Also, by the part (ii) of Lemma 4.5, the hypothesis of Theorem 3.18 are satisfied and the solution of the SDE (4.17) is a diffusion process up to the explosion time τ.

Let show that $P[\tau = \infty] = 1$, using the Feller test for explosions on Theorem 3.12. As $N(t), t \geq 0$ is a diffusion process, the Feller test if satisfied if we prove that $N = 0$ and $N = \infty$ are attracting and unattainable states.

$N = \infty$ is an attracting state. By assumption (F), $g_i(x) \leq A$ for $x > B$. Next, if $\max\{1, B\} < x < c < z < y < a < \infty$,

$$\begin{aligned}
\int_a^x \exp\left[-2\int_c^y \frac{G_i(z)}{\Sigma^2(z)}dz\right] dy &\leq -\int_x^a \exp\left[-\frac{2}{\sigma^2}\int_c^y \frac{g_i(z)}{z}dz\right] dy \\
&\leq -\int_x^a \exp\left[-\frac{2A}{\sigma^2}\ln\left(\frac{y}{c}\right)\right] dy \\
&= -\int_x^a \left(\frac{y}{c}\right)^{-\frac{2A}{\sigma^2}} dy \\
&= -\frac{c^{\frac{2A}{\sigma^2}}}{1-\frac{2A}{\sigma^2}}\left(a^{1-\frac{2A}{\sigma^2}} - x^{1-\frac{2A}{\sigma^2}}\right)
\end{aligned}$$

So, if we take limits in the last expression, we hold that

$$\begin{aligned}
\lim_{a\to\infty} -\frac{c^{\frac{2A}{\sigma^2}}}{1-\frac{2A}{\sigma^2}}\left(a^{1-\frac{2A}{\sigma^2}} - x^{1-\frac{2A}{\sigma^2}}\right) < \infty \;&\Leftrightarrow\; 1 - \frac{2A}{\sigma^2} < 0 \\
&\Leftrightarrow\; \frac{\sigma^2}{2} < A
\end{aligned}$$

Therefore, by Proposition 3.22 $N = \infty$ is an attracting state if and only if $A > \frac{\sigma^2}{2}$.

$N = 0$ is an attracting state. As $G_i(x) \to 0$ as $x \to 0^+$, for each $\epsilon > 0$ there exists $\delta > 0$ such that $G_i(x) < |G_i(x)| < \epsilon$ if $x < \delta$. By this, if $0 < a < c < y <$

$x < \delta < 1$,

$$\begin{aligned}
\int_a^x \exp\left[-2\int_c^y \frac{G_i(z)}{\Sigma^2(z)}dz\right]dy &\leq \int_a^x \exp\left[-2\epsilon\int_c^y \frac{1}{\Sigma^2(z)}dz\right]dy \\
&\leq \int_a^x \exp\left[\frac{2\epsilon}{\sigma^2}\int_c^y dz\right]dy \\
&= \int_a^x \exp\left[\frac{2\epsilon}{\sigma^2}(y-c)\right]dy \\
&= \frac{\sigma^2}{2\epsilon}\left[\exp\left(\frac{2\epsilon}{\sigma^2}(x-c)\right) - \exp\left(\frac{2\epsilon}{\sigma^2}(a-c)\right)\right]
\end{aligned}$$

So, if we take limits in the last expression, we hold that

$$\begin{aligned}
\lim_{a\to 0+}\int_a^x \exp\left[-2\int_c^y \frac{G_i(z)}{\Sigma^2(z)}dz\right]dy &\leq \lim_{a\to 0+}\frac{\sigma^2}{2\epsilon}\left[\exp\left(\frac{2\epsilon}{\sigma^2}(x-c)\right) - \exp\left(\frac{2\epsilon}{\sigma^2}(a-c)\right)\right] \\
&= \frac{\sigma^2}{2\epsilon}\exp\left(-\frac{2\epsilon}{\sigma^2}c\right)\left[\exp\left(\frac{2\epsilon}{\sigma^2}x\right) - 1\right] < \infty
\end{aligned}$$

Therefore, by Lemma 3.20 $N = 0$ is an attracting state.

$N = \infty$ **is an unattainable state.** By assumption (F), $g_i(x) \leq A$ for $x > B$, and choose $\max\{1, B\} < c < z < u < y < a < \infty$. As

$$\begin{aligned}
\int_c^a \frac{S(a) - S(z)}{S'(z)\Sigma^2(z)}dz &= \int_c^a \left(\int_z^a S'(y)\,dy\right)\frac{1}{S'(z)\Sigma^2(z)}dz \\
&= \int_c^a \left(\int_c^y \frac{1}{S'(z)\Sigma^2(z)}dz\right)S'(y)\,dy \\
&= \int_c^a \left(\int_c^y \frac{S'(y)}{S'(z)\Sigma^2(z)}dz\right)dy,
\end{aligned}$$

and

$$\begin{aligned}
\frac{S'(y)}{S'(z)\Sigma^2(z)} &= \frac{1}{\sigma^2 yz}\exp\left[-\int_z^y \frac{2\left(g_i(u) - \frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right] \\
&= \frac{1}{2y\left(g_i(z) - \frac{\sigma^2}{2}\right)}\frac{\partial}{\partial z}\exp\left[\int_y^z \frac{2\left(g_i(u) - \frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right],
\end{aligned}$$

we hold that

$$\begin{aligned}
\lim_{a\to\infty}\int_c^a \frac{S(a) - S(z)}{S'(z)\Sigma^2(z)}dz &= \lim_{a\to\infty}\int_c^a \left(\int_c^y \frac{1}{\sigma^2 yz}\exp\left[-\int_z^y \frac{2\left(g_i(u) - \frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right]dz\right)dy \\
&\geq \frac{1}{\sigma^2}\lim_{a\to\infty}\int_c^a \left(\int_c^y \frac{1}{yz}\exp\left[-\int_z^y \frac{2\left(A - \frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right]dz\right)dy \\
&= \frac{1}{\sigma^2}\lim_{a\to\infty}\int_c^a \left(\int_c^y \frac{1}{yz}\exp\left[-\frac{2\left(A - \frac{\sigma^2}{2}\right)}{\sigma^2}\int_z^y \frac{du}{u}\right]dz\right)dy \\
&= \frac{1}{\sigma^2}\lim_{a\to\infty}\int_c^a \left(\int_c^y \frac{1}{yz}\exp\left[-\frac{2\left(A - \frac{\sigma^2}{2}\right)}{\sigma^2}\ln\left(\frac{y}{z}\right)\right]dz\right)dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sigma^2}\lim_{a\to\infty}\int_c^a\left(\int_c^y \frac{1}{yz}\left(\frac{y}{z}\right)^{-\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}}dz\right)dy \\
&= \frac{1}{\sigma^2}\lim_{a\to\infty}\int_c^a\left(\int_c^y y^{-\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}-1} z^{\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}-1}dz\right)dy \\
&= \frac{1}{2\left(A-\frac{\sigma^2}{2}\right)}\lim_{a\to\infty}\int_c^a y^{-\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}-1}\left(y^{\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}}-c^{\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}}\right)dy \\
&= \frac{1}{2\left(A-\frac{\sigma^2}{2}\right)}\lim_{a\to\infty}\int_c^a \frac{dy}{y}-\frac{c^{\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}}}{2\left(A-\frac{\sigma^2}{2}\right)}\lim_{a\to\infty}\int_c^a y^{-\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}-1}dy \\
&= \frac{1}{2\left(A-\frac{\sigma^2}{2}\right)}\lim_{a\to\infty}\ln\left(\frac{a}{c}\right) \\
&\quad +\frac{\sigma^2 c^{\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}}}{4\left(A-\frac{\sigma^2}{2}\right)^2}\lim_{a\to\infty}\left(a^{-\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}}-c^{-\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}}\right) \\
&= \frac{1}{2\left(A-\frac{\sigma^2}{2}\right)}\lim_{a\to\infty}\ln\left(\frac{a}{c}\right) \\
&\quad +\frac{\sigma^2}{4\left(A-\frac{\sigma^2}{2}\right)^2}\lim_{a\to\infty}\left[\left(\frac{c}{a}\right)^{\frac{2\left(A-\frac{\sigma^2}{2}\right)}{\sigma^2}}-1\right]=\infty
\end{aligned}
$$

if and only is $A > \frac{\sigma^2}{2}$. Therefore, $N = \infty$ is an unattainable state.

$N = 0$ **is an unattainable state.** Let $W = g_i(0^+) - \frac{\sigma^2}{2} > 0$. In this case, for each $\alpha > 0$ there exists $\delta > 0$ such that $g_i(x) - \frac{\sigma^2}{2} \geq -\alpha$, for $x \in (0,\delta)$. Let choose $a, c, u, y, z \in (0,\delta)$ such that $0 < a < y < u < z < c < \delta$. Then, $g_i(u) - \frac{\sigma^2}{2} \geq -\alpha$ for $u \in (y, z)$ and

$$
\begin{aligned}
\lim_{a\to0^+}\int_a^c \frac{S(a)-S(z)}{S'(z)\Sigma^2(z)}dz &= \frac{1}{\sigma^2}\lim_{a\to0^+}\int_a^c\left(\int_a^z\frac{1}{yz}\exp\left[\int_y^z\frac{2\left(g_i(u)-\frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right]dy\right)dz \\
&\geq \frac{1}{\sigma^2}\lim_{a\to0^+}\int_a^c\left(\int_a^z\frac{1}{yz}\exp\left[\int_y^z\frac{-2\alpha}{\sigma^2 u}du\right]dy\right)dz \\
&= \frac{1}{\sigma^2}\lim_{a\to0^+}\int_a^c\left(\int_a^z\frac{1}{yz}\exp\left[\frac{-2\alpha}{\sigma^2}\int_y^z\frac{du}{u}\right]dy\right)dz \\
&= \frac{1}{\sigma^2}\lim_{a\to0^+}\int_a^c\left(\int_a^z\frac{1}{yz}\exp\left[\frac{-2\alpha}{\sigma^2}\ln\left(\frac{z}{y}\right)\right]dy\right)dz \\
&= \frac{1}{\sigma^2}\lim_{a\to0^+}\int_a^c\left(\int_a^z\frac{1}{yz}\left(\frac{z}{y}\right)^{\frac{-2\alpha}{\sigma^2}}dy\right)dz \\
&= \frac{1}{\sigma^2}\lim_{a\to0^+}\int_a^c z^{-\frac{2\alpha}{\sigma^2}-1}\left(\int_a^z y^{\frac{2\alpha}{\sigma^2}-1}dy\right)dz \\
&= \frac{1}{2\alpha}\lim_{a\to0^+}\int_a^c z^{-\frac{2\alpha}{\sigma^2}-1}\left(z^{\frac{2\alpha}{\sigma^2}}-a^{\frac{2\alpha}{\sigma^2}}\right)dz
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\alpha}\lim_{a\to 0^+}\int_a^c \left(z^{-1}-a^{\frac{2\alpha}{\sigma^2}}z^{-\frac{2\alpha}{\sigma^2}-1}\right)dz \\
&= \frac{1}{2\alpha}\lim_{a\to 0^+}\left(\left[\ln(c)-\ln(a)\right]+\frac{\sigma^2}{2\alpha}\left[\left(\frac{a}{c}\right)^{\frac{2\alpha}{\sigma^2}}-1\right]\right)=\infty.
\end{aligned}
$$

Now, suppose $W<0$. By Lemma 4.6, there exists $D,\beta>0$ such that (4.25) is satisfied for $x<\delta$. Then, choosing $a,c,u,y,z\in(0,\delta)$ such that $0<a<y<u<z<c<\delta$, we have that

$$
\begin{aligned}
\lim_{a\to 0^+}\int_a^c \frac{S(a)-S(z)}{S'(z)\Sigma^2(z)}dz &= \lim_{a\to 0^+}\int_a^c\left(\int_y^c \frac{1}{2y\left(g_i(z)-\frac{\sigma^2}{2}\right)}\frac{\partial}{\partial z}\exp\left[\int_y^z \frac{2\left(g_i(u)-\frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right]dz\right)dy \\
&\geq \frac{1}{2\left(D+\frac{\sigma^2}{2}\right)}\lim_{a\to 0^+}\int_a^c\left(\int_y^c \frac{1}{y\ln(z)}\frac{\partial}{\partial z}\exp\left[\int_y^z \frac{2\left(g_i(u)-\frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right]dz\right)dy \\
&\geq \frac{1}{2\left(D+\frac{\sigma^2}{2}\right)}\lim_{a\to 0^+}\int_a^c\left(\int_y^c \frac{1}{y\ln(y)}\frac{\partial}{\partial z}\exp\left[\int_y^z \frac{2\left(g_i(u)-\frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right]dz\right)dy \\
&= \frac{1}{2\left(D+\frac{\sigma^2}{2}\right)}\lim_{a\to 0^+}\int_a^c\frac{1}{y\ln(y)}\left(\int_y^c \frac{\partial}{\partial z}\exp\left[\int_y^z \frac{2\left(g_i(u)-\frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right]dz\right)dy \\
&= \frac{1}{2\left(D+\frac{\sigma^2}{2}\right)}\lim_{a\to 0^+}\int_a^c\frac{1}{-y\ln(y)}\left(1-\exp\left[\int_y^c \frac{2\left(g_i(u)-\frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right]\right)dy \\
&\geq \frac{1}{2\left(D+\frac{\sigma^2}{2}\right)}\lim_{a\to 0^+}\int_a^c\frac{1}{-y\ln(y)}\left(1-\exp\left[\int_y^c \frac{-2\beta}{\sigma^2 u}du\right]\right)dy \\
&= \frac{1}{2\left(D+\frac{\sigma^2}{2}\right)}\lim_{a\to 0^+}\int_a^c\frac{1}{-y\ln(y)}\left(1-\left(\frac{c}{y}\right)^{-\frac{2\beta}{\sigma^2}}\right)dy \\
&= \frac{1}{2\left(D+\frac{\sigma^2}{2}\right)}\lim_{a\to 0^+}\int_a^c\frac{1}{-y\ln(y)}dy \\
&\quad -\frac{1}{2\left(D+\frac{\sigma^2}{2}\right)}\lim_{a\to 0^+}\int_a^c\frac{1}{-y\ln(y)}\left(\frac{c}{y}\right)^{-\frac{2\beta}{\sigma^2}}dy=\infty
\end{aligned}
$$

because that

$$
\begin{aligned}
\lim_{a\to 0^+}\int_a^c \frac{1}{-y\ln(y)}\left(\frac{c}{y}\right)^{-\frac{2\beta}{\sigma^2}}dy &\leq \lim_{a\to 0^+}\frac{c^{\frac{-2\beta}{\sigma^2}}}{-\ln(c)}\int_a^c y^{\frac{2\beta}{\sigma^2}-1}dy \\
&= \frac{\sigma^2}{-2\beta\ln(c)}\lim_{a\to 0^+}\left[1-\left(\frac{a}{c}\right)^{\frac{2\beta}{\sigma^2}}\right] \\
&= \frac{\sigma^2}{-2\beta\ln(c)}<\infty
\end{aligned}
$$

and that

$$
\lim_{a\to 0^+}\int_a^c \frac{1}{-y\ln(y)}dy=\lim_{a\to 0^+}\ln\left(\frac{-\ln(a)}{-\ln(c)}\right)=\infty. \tag{4.26}
$$

Therefore, $N=0$ is an unattainable state, and assumption (C) is satisfied and it implies, by the Feller test, that $P[\tau=\infty]=1$.

As $N(t), t \in (0,\infty)$ is a diffusion process on $(0,\infty)$, then assumption (D) is satisfied and thus, $R_a(x)$ exists and is equal to $g_i(x)$. Now we proceed to show that assumption (E) is satisfied.

Let consider the process $Y(t), t \in (0,\infty)$, with $Y(t) = \ln(N(t))$. By the Itô formula (2.19), this process is a solution of the SDE (4.20)

$$dY(t) = \left(g_i\left(e^{Y(t)}\right) - \frac{\sigma^2}{2}\right) dt + \sigma dB(t)$$

with initial condition $Y(0) = \ln(N_0)$. As $g_i(x)$ satisfies (iii) and (iv) of Lemma 4.5, then $\bar{G}_i(y) := g_i(e^y) - \frac{\sigma^2}{2}$ and $\bar{\Sigma}(x) := \sigma$ fulfilled the Itô conditions 2; this implies that the solution exists up to an explosion time $\bar{\tau}$. As $N(t), t \in (0,\infty)$ is a diffusion process, the Theorem 3.17 implies that $Y(t), t \in \mathbb{R}$ is also a diffusion process. Moreover, as $N = 0$ and $N = \infty$ are unattainable states, Proposition 3.27 implies that $Y = \ln(0) = -\infty$ and $Y = \ln(\infty) = \infty$ are unattainable states. This implies that the explosion time $\bar{\tau}$ for this process is also ∞. Thus, assumption (E) is also satisfied and so, $R_g(x) = g_i(x) - \frac{\sigma^2}{2}$. □

The Stratonovich case

Let us consider the Stratonovich SDE

$$dN(t) = g_s(N(t))\,N(t)\,dt + \sigma N(t) \circ dB(t), \tag{4.27}$$

with initial condition $N(0) = N_0 > 0$ and let $G_s(x) := x g_s(x)$ and $\Sigma(x) := \sigma x$. The previous analysis realized to the Itô SDE (4.17) can be applied to the Stratonovich SDE (4.27) if we works with its equivalent Itô SDE

$$dN(t) = g_i(N(t))\,N(t)\,dt + \sigma N(t)\,dB(t), \tag{4.28}$$

where $g_i(x) := g_s(x) + \frac{\sigma^2}{2}$.

In order to satisfies assumptions (A-H), we assume that

(A') $g_s : (0,\infty) \to \mathbb{R}$ is a continuously differentiable function;

(B') the limits $g_s(0^+) := \lim_{N\to 0^+} g_s(N)$, $G_s(0^+) := \lim_{N\to 0^+} G_s(N)$ exists and, independent of the value of $g_s(0^+)$, $G_s(0^+) = 0$.

(C') $N = 0$ and $N = \infty$ are unattainable states of the process $N(t), t \geq 0$;

(D') the limits

$$\lim_{h\to 0^+} \frac{\mathbb{E}\left[N(t+h) - x \middle| N(t) = x\right]}{h}; \quad \lim_{h\to 0^+} \frac{\mathbb{E}\left[|N(t+h) - x|^2 \middle| N(t) = x\right]}{h} \tag{4.29}$$

exists;

(E') the limits

$$\lim_{h\to 0^+}\frac{\mathbb{E}\left[\ln\left(N\left(t+h\right)\right)-\ln\left(x\right)\middle|N\left(t\right)=x\right]}{h};\ \lim_{h\to 0^+}\frac{\mathbb{E}\left[\left|\ln\left(N\left(t+h\right)\right)-\ln\left(x\right)\right|^2\middle|N\left(t\right)=x\right]}{h} \tag{4.30}$$

exists;

(F') There are $A\in(-\infty,\infty)$ and $B\in(0,\infty)$ such that $g_s(x)\leq A$ for $x>B$.

(G') If it happens that $g_s(0^+)=\infty$, then we must have $g_s(x)\leq C\left(1+\frac{1}{x^2}\right)$ (with C some positive finite constant) for $x\in(0,\delta)$ (with $\delta>0$).

(H') If it happens that $g_s(0^+)=-\infty$, then we must have $g_s(x)\geq D\ln(x)$ (with D some positive finite constant) for $x\in(0,\delta)$ (with $\delta>0$).

We hold the next observations:

- assumptions (A'), (B') and (C') guarantee the existence of solution $N(t), t\geq 0$ for the Stratonovich SDE (4.27);
- (D') and (E') provides the existence of the average growth rates;
- if (F'), (G'), (H') are satisfied, then these implies that (C'), (D') and (E') hold true.

Therefore, we hold the next theorems, for existence and uniqueness of solution for the Stratonovich SDE (4.27) and the existence of the growth rates.

Theorem 4.8. *Let $g_s(x)$ be a function that satisfies assumptions (A') and (B') and let $N(t)$ be the solution of the Stratonovich SDE*

$$dN(t)=g_s(N(t))N(t)\,dt+\sigma N(t)\circ dB(t)$$

If assumption (C') is satisfied, then $N(t)$ exists for all $t\geq 0$. Moreover,

- *if assumption (D') is satisfied, the arithmetic growth rate exists and $R_a(x)=g_s(x)+\frac{\sigma^2}{2}$.*
- *if assumption (E') is satisfied, the geometric growth rate exists and $R_g(x)=g_s(x)$.*

Theorem 4.9. *Let $N(t)$ be the solution of the Stratonovich SDE*

$$dN(t)=g_s(N(t))N(t)\,dt+\sigma N(t)\circ dB(t)$$

with initial condition $N(0)=N_0>0$. If assumptions (A'), (B'), (F'), (G') and (H') are satisfies, then assumptions (C'), (D') and (E') are satisfied and so, by Theorem 4.8:

- *$N(t)$ exists for all $t\geq 0$ and it is a diffusion process,*

- *the arithmetic growth rate* $R_a(x)$ *exists and* $R_a(x) = g_s(x) + \frac{\sigma^2}{2}$,
- *the geometric growth rate* $R_g(x)$ *exists and* $R_g(x) = g_s(x)$.

Comparison between both solution. Let $N_i(t), t \geq 0$ be solution of the SDE (4.17) and $N_s(t), t \geq 0$ be solution of the SDE (4.27). If Theorems 4.7 and 4.9 are satisfied, the $N_i(t)$ and $N_s(t)$ are diffusion processes, where the drift coefficients are

$$A_i(x) = xg_i(x), \quad A_s(x) = x\left(g_s(x) + \frac{\sigma^2}{2}\right) \tag{4.31}$$

and both have the same diffusion coefficient $B(x) = \sigma^2 x^2$. If we observe that

$$R_a(x) = g_i(x)$$

for the Itô case and

$$R_a(x) = g_s(x) + \frac{\sigma^2}{2}$$

in the Stratonovich case, then the drift coefficient $A_i(x) = xR_a(x)$ and $A_s(x) = xR_a(x)$. Therefore, both solutions are the same diffusion process and thus implies that Itô SDE (4.17) and Stratonovich SDE (4.27) have exactly the same solution in terms of their specified average growth rate.

4.4 The resolution of the controversy: the harvesting case

Let us extend the above analysis to the harvesting case. For that purpose, consider a per-capita growth rate $r(t)$ with *harvesting effort* $h(t)$,

$$r(t) = g(t) - h(t) + \sigma(t) W(t). \tag{4.32}$$

In this per-capita growth rate, let assume that g and h are functions the current population size $g(t) := g(N(t))$, $h(t) := h(N(t))$, that $g(N)$ is a twice continuously differentiable function with $\frac{dg(N)}{dN} < 0$ for all $N > 0$, $g(\infty) := \lim_{N\to\infty} g(N) < 0$ and $G(0^+) := \lim_{N\to 0} Ng(N) = 0$ and that $h(x)$ is also a non-negative twice continuously differentiable function with $H(0^+) := \lim_{N\to 0} Nh(N) = 0$. Also, assume that $|\frac{\sigma}{g(x)}|$ is bounded in a right neighborhood of $N = 0$.

The Itô case

Let denote as $g_i(x)$ the average growth rate. Under the above assumptions, the Itô stochastic differential equation that governs the model is

$$dN(t) = N(t)\left[g_i(N(t)) - h(N(t))\right]dt + \sigma N(t)\, dB(t), \tag{4.33}$$

with initial condition $N(0) = N_0 > 0$; this SDE has a global solution $N(t), t \geq 0$ that exists up to an time-explosion τ, given by Theorem 3.2.

To avoid this phenomena and obtain the existence and uniqueness of the solution $N(t), t \geq 0$ for this SDE, let define the *harvesting rate* $\mathsf{H}(N)$ such that

$$\mathsf{H}(N) = \frac{h(N)}{g_i(N)}$$

and further assume that

(I) if $\mathsf{H}(0^+) < 1$, then there exists $\delta, C > 0$ such that $g_i(x) - h(x) \leq C\left(1 + \frac{1}{x^2}\right)$ for all $x < \delta$;

(II) if $\mathsf{H}(0^+) > 1$, then there exists $\delta, D > 0$ such that $g_i(x) - h(x) \geq D \ln(x)$ for all $x < \delta$.

This dichotomy about $\mathsf{H}(0^+)$ substituted assumptions (G) and (H) from Section 4.3. The next lemma is analogous to Lemma 4.5.

Lemma 4.10. *Assume that $g_i(x), h(x)$ are two continuously differentiable function as in the above. Then, for all $x > 0$.*

(i) there exists a constant $L_1 > 0$ such that

$$x^2 (g_i(x) - h(x)) \leq L_1 \left(1 + x^2\right); \tag{4.34}$$

(ii) there exists a constant $L_2 > 0$ such that

$$x (g_i(x) - h(x)) \leq L_2 \left(1 + x^2\right); \tag{4.35}$$

(iii) if also assumption (II) is satisfied, there exists a constant $L_3 > 0$ such that

$$\ln(x) \left(g_i(x) - h(x) - \frac{\sigma^2}{2}\right) \leq L_3 \left(1 + \ln(x)^2\right); \tag{4.36}$$

(iv) if also assumption (I) is satisfied, there exists a constant $L_4 > 0$ such that

$$g_i(x) - h(x) - \frac{\sigma^2}{2} \leq L_4 \left(1 + \ln(x)^2\right). \tag{4.37}$$

The proof of the above lemma is similar to the proof of Lemma 4.5. The principal difference is the use of $\mathsf{H}(0^+)$ indeed $g_i(0^+)$. With this lemma, we justify the existence and uniqueness of a solution for the SDE (4.33).

Theorem 4.11. *Let $g_i(x), h(x)$ be two continuously differentiable functions as above and that fulfilled assumptions (I) and (II). Then, there exists a unique diffusion process $N(t), t \geq 0$ that is solution of the Itô SDE*

$$dN(t) = N(t)(g_i(N(t)) - h(N(t)))\,dt + \sigma N(t)\,dB(t) \tag{4.38}$$

with initial condition $N(0) = N_0 > 0$, and implies that:

- *the states $N = 0$ and $N = \infty$ are unattainable, independent of the value of* $\mathsf{H}(0^+)$*,*
- *the arithmetic growth rate $R_a(x)$ exists and $R_a(x) = g_i(x) - h(x)$,*
- *the geometric growth rate $R_g(x)$ exists and $R_g(x) = g_i(x) - h(x) - \frac{\sigma^2}{2}$.*

Proof. The existence and uniqueness of a solution $N(t), t \geq 0$ for the SDE (4.38) up to an time-explosion τ is fulfilled by Lemma 4.10 (i). This solution is a diffusion by Lemma 4.10 and Theorem 3.28. The proof of that the states $N = 0$ and $N = \infty$ are unattainable is the same as in Theorem 4.7, using assumptions (I) and (II) indeed assumptions (G) and (H) of Section 4.3.

Recall $G_i(x) = xg_i(x)$ and $\Sigma(x) = \sigma x$. If $\mathsf{H}(0^+) > 1$, then there exists $A_1 > 0$ such that $1 - \mathsf{H}(x) \leq -A$ for all x in a right neighborhood R_1 of $N = 0$. Then, for $z < x < y < \delta$,

$$\begin{aligned}
\frac{(1-\mathsf{H}(x))\,G_i(x)}{\Sigma(x)^2} \leq \frac{-AG_i(x)}{\Sigma(x)^2} \quad &\Rightarrow \quad -2\frac{(1-\mathsf{H}(x))\,G_i(x)}{\Sigma(x)^2} \geq 2A\frac{G_i(x)}{\Sigma(x)^2} \\
&\Rightarrow \quad -2\int_z^y \frac{(1-\mathsf{H}(x))\,G_i(x)}{\Sigma(x)^2}dx \geq 2A\int_z^y \frac{G_i(x)}{\Sigma(x)^2}dx \\
&\Rightarrow \quad 2A\int_y^z \frac{G_i(x)}{\Sigma(x)^2}dx \geq -2\int_y^z \frac{(1-\mathsf{H}(x))\,G_i(x)}{\Sigma(x)^2}dx
\end{aligned}$$

This implies that, for $0 < a < z < x < y < b < \delta$,

$$\begin{aligned}
\exp\left[-2\int_y^z \frac{(1-\mathsf{H}(x))\,G_i(x)}{\Sigma(x)^2}dx\right] &\leq \exp\left[2A\int_y^z \frac{G_i(x)}{\Sigma(x)^2}dx\right] \\
&\leq \frac{y}{z}\exp\left[2A\int_y^z \frac{G_i(x)}{\Sigma(x)^2}dx\right] \\
&= \left(\frac{\Sigma(y)}{2A}\right)\left(\frac{\Sigma(z)}{G_i(z)}\right)\left(\frac{2AG_i(z)}{\Sigma(z)^2}\right)\exp\left[2A\int_y^z \frac{G_i(x)}{\Sigma(x)^2}dx\right] \\
&= \left(\frac{\Sigma(y)}{2A}\right)\left(\frac{\Sigma(z)}{G_i(z)}\right)\frac{d}{dz}\exp\left[2A\int_y^z \frac{G_i(x)}{\Sigma(x)^2}dx\right] \\
&\leq \frac{C\Sigma(y)}{2A}\frac{d}{dz}\exp\left[2A\int_y^z \frac{G(x)}{\Sigma(x)^2}dx\right]
\end{aligned}$$

and that

$$\begin{aligned}
\int_a^b \exp\left[-2\int_y^z \frac{(1-\mathsf{H}(x))\,G_i(x)}{\Sigma(x)^2}dx\right]dz &\leq \frac{C\Sigma(y)}{2A}\int_a^b \frac{d}{dz}\exp\left[2A\int_y^z \frac{G_i(x)}{\Sigma(x)^2}dx\right]dz \\
&= \frac{C\Sigma(y)}{2A}\left(\exp\left[2A\int_y^b \frac{G_i(x)}{\Sigma(x)^2}dx\right]\right. \\
&\quad \left. - \exp\left[-2A\int_a^y \frac{G_i(x)}{\Sigma(x)^2}dx\right]\right).
\end{aligned}$$

Taking limits in both sides, it holds that

$$\begin{aligned}
\lim_{a\to 0^+}\int_a^b \exp\left[-2\int_y^z \frac{(1-\mathsf{H}(x))\,G_i(x)}{\Sigma(x)^2}dx\right]dz &\leq \frac{C\Sigma(y)}{2A}\left(\exp\left[2A\int_y^b \frac{G_i(x)}{\Sigma(x)^2}dx\right]\right. \\
&\quad \left. - \lim_{a\to 0^+}\exp\left[-2A\int_a^y \frac{G_i(x)}{\Sigma(x)^2}dx\right]\right) < +\infty
\end{aligned}$$

because the integral inside the first exponential is finite and the integral inside the second one is finite or $-\infty$. This implies that $N=0$ is a non-attracting state, and also, an unattainable state by Theorem 3.24.

In the case that $\mathsf{H}(0^+)<1$, the state $N=0$ is attracting, and we omit the proof because is the same as in Theorem 4.7. In this case, we must show that $N=0$ is unattainable, that is similar part of the proof of Theorem 4.7 for $W>0$. In fact, as $\mathsf{H}(0^+)<1$, there exists $\delta>0$ such that $1-\mathsf{H}(x)\geq 0$ for all $x<\delta$. Then, $g_i(x)-h(x)\geq 0$ for all $x<\delta$, implying that there exists $\alpha>0$ such that

$$g_i(x)-h(x)-\frac{\sigma^2}{2}\geq -\alpha.$$

Let choose $a,c,u,y,z\in(0,\delta)$ such that $0<a<y<u<z<c<\delta$. Then, $g_i(u)-\frac{\sigma^2}{2}\geq -\alpha$ for $u\in(y,z)$ and

$$\begin{aligned}
\lim_{a\to 0^+}\int_a^c \frac{S(a)-S(z)}{S'(z)\Sigma^2(z)}dz &= \frac{1}{\sigma^2}\lim_{a\to 0^+}\int_a^c\left(\int_a^z \frac{1}{yz}\exp\left[\int_y^z \frac{2\left(g_i(u)-h(u)-\frac{\sigma^2}{2}\right)}{\sigma^2 u}du\right]dy\right)dz\\
&\geq \frac{1}{\sigma^2}\lim_{a\to 0^+}\int_a^c\left(\int_a^z \frac{1}{yz}\exp\left[\int_y^z \frac{-2\alpha}{\sigma^2 u}du\right]dy\right)dz\\
&= \frac{1}{\sigma^2}\lim_{a\to 0^+}\int_a^c\left(\int_a^z \frac{1}{yz}\exp\left[\frac{-2\alpha}{\sigma^2}\int_y^z \frac{du}{u}\right]dy\right)dz\\
&= \frac{1}{\sigma^2}\lim_{a\to 0^+}\int_a^c\left(\int_a^z \frac{1}{yz}\exp\left[\frac{-2\alpha}{\sigma^2}\ln\left(\frac{z}{y}\right)\right]dy\right)dz\\
&= \frac{1}{\sigma^2}\lim_{a\to 0^+}\int_a^c\left(\int_a^z \frac{1}{yz}\left(\frac{z}{y}\right)^{\frac{-2\alpha}{\sigma^2}}dy\right)dz\\
&= \frac{1}{\sigma^2}\lim_{a\to 0^+}\int_a^c z^{-\frac{2\alpha}{\sigma^2}-1}\left(\int_a^z y^{\frac{2\alpha}{\sigma^2}-1}dy\right)dz\\
&= \frac{1}{2\alpha}\lim_{a\to 0^+}\int_a^c z^{-\frac{2\alpha}{\sigma^2}-1}\left(z^{\frac{2\alpha}{\sigma^2}}-a^{\frac{2\alpha}{\sigma^2}}\right)dz\\
&= \frac{1}{2\alpha}\lim_{a\to 0^+}\int_a^c\left(z^{-1}-a^{\frac{2\alpha}{\sigma^2}}z^{-\frac{2\alpha}{\sigma^2}-1}\right)dz\\
&= \frac{1}{2\alpha}\lim_{a\to 0^+}\left(\left[\ln(c)-\ln(a)\right]+\frac{\sigma^2}{2\alpha}\left[\left(\frac{a}{c}\right)^{\frac{2\alpha}{\sigma^2}}-1\right]\right)=\infty,
\end{aligned}$$

as we need to show. With this, $N=0$ and $N=\infty$ are unattainable states. Thus, by the Feller test for explosions, $P[\tau=\infty]=1$. Then, there is no explosion, implying that the arithmetic average growth rate $R_a(x)$ exists.

The proof of the geometrical average growth rate $R_g(x)$ is the same as in Theorem 4.7. □

The Stratonovich case

By other hand, let consider the Stratonovich differential equation that governs the model (4.32)

$$dN(t)=N(t)\left[g_s(N(t))-h(N(t))\right]dt+\sigma N(t)\circ dB(t), \tag{4.39}$$

with initial condition $N(0) = N_0 > 0$ and suppose that $g_s(x), h(x)$ and $\sigma > 0$ satisfies the previous hypothesis at the beginning of the section, changing g by g_s. This SDE is equivalent to the Itô SDE

$$dN(t) = N(t)\left[g_i(N(t)) - h(N(t)) + \frac{\sigma^2}{2}\right] dt + \sigma N(t)\, dB(t),$$

that has a global solution $N(t), t \geq 0$ that exists up to an time-explosion τ, given by Theorem 3.2. The assumptions (I) and (II) can be adequate to these ones:

(I') if $\mathsf{H}(0^+) < 1$, then there exists $\delta, C > 0$ such that $g_s(x) - h(x) \leq C\left(1 + \frac{1}{x^2}\right)$ for all $x < \delta$;

(II') if $\mathsf{H}(0^+) > 1$, then there exists $\delta, D > 0$ such that $g_s(x) - h(x) \geq D \ln(x)$ for all $x < \delta$.

With this, we hold the next theorem,

Theorem 4.12. *Let $g_s(x), h(x)$ be two continuously differentiable functions as above and that fulfilled assumptions (I') and (II'). Then, there exists a unique diffusion process $N(t), t \geq 0$ that is solution of the Itô SDE*

$$dN(t) = N(t)(g_s(N(t)) - h(N(t)))\, dt + \sigma \circ N(t)\, dB(t) \tag{4.40}$$

with initial condition $N(0) = N_0 > 0$, and implies that:

- *the states $N = 0$ and $N = \infty$ are unattainable, independent of the value of $\mathsf{H}(0^+)$,*
- *the arithmetic growth rate $R_a(x)$ exists and $R_a(x) = g_s(x) - h(x) + \frac{\sigma^2}{2}$,*
- *the geometric growth rate $R_g(x)$ exists and $R_g(x) = g_s(x) - h(x)$.*

Comparison between both solution. Let $N_i(t), t \geq 0$ be solution of the SDE (4.38) and $N_s(t), t \geq 0$ be solution of the SDE (4.40). If Theorems 4.11 and 4.12 are satisfied, the $N_i(t)$ and $N_s(t)$ are diffusion processes, where the drift coefficients are

$$A_i(x) = x g_i(x), \quad A_s(x) = x\left(g_s(x) + \frac{\sigma^2}{2}\right) \tag{4.41}$$

and both have the same diffusion coefficient $B(x) = \sigma^2 x^2$. If we observe that

$$R_a(x) = g(x)$$

for the Itô case and

$$R_a(x) = g_s(x) + \frac{\sigma^2}{2}$$

in the Stratonovich case, then the drift coefficient $A_i(x) = xR_a(x)$ and $A_s(x) = xR_a(x)$. Therefore, both solutions are the same diffusion process and thus implies that Itô SDE (4.17) and Stratonovich SDE (4.27) have exactly the same solution in terms of their specified average growth rate.

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